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PARALLEL PREFERENCE STRUCTURES IN LABOR SUPPLY AND COMMODITY DEMAND: AN ADAPTATION OF THE GORMAN POLAR FORM

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AN ADAPTATION OF THE GORMAN POLAR FORM

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ABSTRACT

Parallel preference structures are characterized by indifference surfaces that are identical in shape and scale, each being a translation of a basic surface along parallel income-consumption curves. The purpose of this paper is to discuss the properties of parallel structures and their potential usefulness in models of labor supply and commodity demand. Limited applications in production analysis are also discussed but are not the primary focus of the paper.

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INTRODUCTION

Parallel preference structures are characterized by indifference surfaces that are identical in shape and scale, each being a translation of a basic surface along parallel income-consumption curves. The purpose of this paper is to discuss the properties of parallel structures and their potential usefulness in models of labor supply and commodity demand. Limited applications in production analysis are also discussed but are not the primary focus of the paper.

In their most tractable form, with linear income-consumption curves, parallel structures are a special case of the Gorman Polar Form [14, 15].[1] A suitably parameterized cost or expenditure function for a linear parallel structure provides a second order point approximation to an arbitrary general cost or expenditure function. By that criterion, a variety of simple versions of the parallel structure are on roughly equal footing with other flexible functional forms employed in recent demand, production, and labor supply research (see, for instance, Christensen and Greene [5], Christensen, Jorgenson, and Lau [6], Christensen, Jorgensen, and Lau [7], and Wales and Woodland [22]). Parallel structures are quite distinctive, however, in their global properties which render them potentially very useful in some applications and patently inappropriate in others. Within the domain of potentially suitable applications a variety of forms of the parallel structure may be employed to tailor the model to the requirements of a particular problem.

The most distinctive global property of parallel structures is that the substitution characteristics are the same, in absolute magnitude, at all levels of utility or of production. In a production context, it is unlikely that such a structure would accurately represent technology at three units of output and at three million. By contrast, for a labor supply model, in which the time endowment is fixed regardless of utility level, the parallel model may provide a very good representation of income-leisure preferences. Approximate versions of the linear parallel structure have in fact been employed in two labor supply studies by Ashenfelter and Heckman [1, 2]. The generally tenable results of those studies provide encouragement for further implementation of parallel structures in labor supply research. Interpretation of the Ashenfelter-Heckman models in terms of the parallel model also indicates a need for slight revisions in their interpretation of parameters and in the resulting estimation restrictions.

The constancy of absolute substitution characteristics is clearly a restrictive feature of parallel structures, but it is a source of flexibility in other dimensions. The substitution characteristics are parameterized as a separable portion of the functional form and can easily be adapted to the needs of a particular problem. For this purpose, numerous second order parameterizations are available, each with different global substitution properties. Yet more general functional forms may also be employed if necessary. This flexibility in the parameterization of substitution effects recommends the parallel model for applications in which there are large variations in relative prices, a characteristic frequently encountered in labor supply models.

In other applications, nonlinear versions of the parallel structure can model flexible income effects while maintaining independent flexibility of substitution characteristics. For example, nonlinear parallel structures provide possible models for the study of demand for goods that are normal and inferior in different income ranges.

The organization of the remainder of the paper is as follows. In section I, the functional form of the General Parallel Structure is presented. Linear parallel structures are presented as a special case and a variety of specific forms are shown to be second order point approximations to the cost function for an arbitrary structure. The more extended substitution characteristics of the alternative forms are discussed and compared. Estimation forms for linear and nonlinear parallel structures are presented in section II. The interpretation of the Ashenfelter-Heckman model in terms of the parallel model is presented in section III together with a brief discussion of applications of the parallel model to labor supply. A brief summary and concluding remarks are presented in section IV.

I. PARAMETERIZATION OF PARALLEL STRUCTURES

A general parallel preference structure may be parameterized in terms of its corresponding expenditure or cost function in the form

$$(1) \qquad C(u, p) = \sum_{i=1}^{N} p_i f_i(u) + \Lambda(p)$$

where u is a utility index and p is the N x 1 vector of prices. The cost function must be concave and positively linear homogeneous (PLH) with respect to prices and increasing in u.[2] These conditions are always fulfilled if $\Lambda(p)$ is a concave PLH function, and the $f_i(u)$ are continuous positive nondecreasing functions.[3] Some of the $f_i(u)$ may be decreasing functions, corresponding to inferior goods, but in such cases $C(u,p)$ will be increasing in u only in the price domain for which $\sum_{i=1}^{N} p_i f_i'(u) > 0$, assuming that the f_i are differentiable.

The characteristics of parallel preference structures are more easily visualized by considering the corresponding vector of Hicksian demand functions (or derived demand functions). As shown by Hicks [16] and noted by Hotelling [17] in a production context, these functions are simply the price derivatives of the cost function.

$$(2) \qquad X = \Phi(u, p) = \nabla_p C(u, p) = f(u) + \Psi(p)$$

The gradient vector $\nabla_p \Lambda(p) = \Psi(p)$, with elements $\psi^i(p)$, defines the compensated unit demand functions that are identical at all levels of utility but for systematic translation through consumption space. This set of demand functions may also be thought of as providing a parametric definition of the basic indifference surface. The vector function $f(u)$, with elements $f_i(u)$, parametrically defines the basic income-consumption curve (ICC) that is the locus of reference points for successive sets of unit demand functions, e.g., the ICC for prices p_o at which $\Psi(p_o) = \underline{0}$. The ICC for any other set of prices is parallel to the base curve, displaced by the vector $\Psi(p)$. A simple two-dimensional example is illustrated in Figure 1. Note that some (u, p) values may imply optimal consumption points outside the positive orthant and thus correspond to corner solutions for one or more goods. Discussion in this paper is limited to interior solutions.

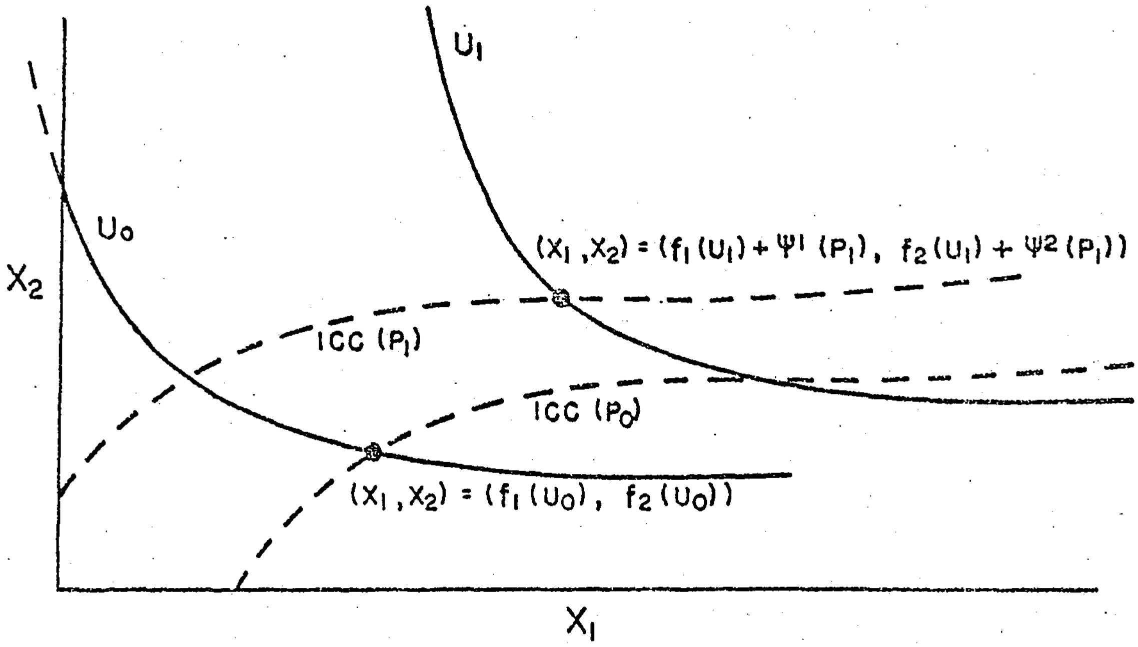

FIGURE 1: A Parallel Structure

If deemed appropriate in a production context, the derived demand functions (2), with output, Q, substituted for u, may be parameterized and estimated directly.[4] Further transformations are necessary if the nonlinear parallel preference model is to be estimable in a consumption context in which u is not observable. We return to the latter case below after considering the more tractable linear form.

Linear parallel structures have linear income-consumption curves parallel to a basic ICC defined by the functions $f_i(u) = \delta_i h(u)$. The expenditure/cost function has the form (3).

$$(3) \qquad \mathbb{C}(u, p) = h(u) \sum_{i=1}^{N} \delta_i p_i + \Lambda(p)$$

This is a special case of the cost function for the Gorman Polar Form (4).

$$(4) \qquad \mathbb{C}(u, p) = h(u)\,\Pi(p) + \Lambda(p)$$

The Gorman Polar Form (4), with $\Pi(p)$ taken to be any concave PLH function, is the most general formulation of models having linear income-consumption curves. It is also the most general model of individual preferences that yields a globally consistent aggregate preference function that is independent of the distribution of income. Blackorby, Boyce, and Russell [3] discuss several special cases of the Gorman Polar Form. They characterize linear parallel structures as "Homothetic to Minus Infinity," following Pollak [20] and Chipman [4].

The linear parallel expenditure function may easily be parameterized to provide a second order approximation to an arbitrary general cost/expenditure function. Such an approximation, in the Diewert sense of matching first and second derivatives at the approximation point, follows from the choice of suitably flexible forms for $\Lambda(p)$ and $h(u)$. The restriction that $\Lambda(p)$ must be linearly homogeneous rules out direct use of forms, such as the quadratic or transcendental logarithmic functions, that are based on Taylor expansions.[5] A convenient modification that permits the use of this class of flexible forms involves simple deflation and premultiplication by the price of a numeraire good. The deflated unit cost function may then be expressed as a flexible nonhomogeneous function of the remaining N-1 normalized prices, $p_i^* = p_i/p_1$, where the first good is chosen as numeraire.

The general form for the normalized second order parameterization of the Linear Parallel model, with the first good as numeraire, is shown in Table 1 together with its first and second derivatives. The $T^i(p_i^*)$ represent increasing concave transformations of p_i^* with continuous derivatives denoted T_i^i and T_{ii}^i. (Arguments are omitted to conserve space.) The β_{ij} are symmetric parameters and the functional notation, $\phi^i(u, p)$ and $s_{ij}(p)$, serves to facilitate subsequent discussion of the Hicksian demand functions and the substitution effects respectively. The general cost function is PLH in nominal prices and is concave in the domain for which the Hessian matrix with respect to prices is negative semidefinite.

At any given approximation point a general cost function may have arbitrary first partials (the price derivatives determining the level under linear homogeneity) and arbitrary second partials involving u. Homogeneity and symmetry restrictions leave N(N-1)/2 arbitrary second partials with respect to price. The parameterization T1.1 accommodates any set of such values. The N+2 first and second partials involving u,

TABLE 1

NORMALIZED GENERAL TAYLOR SERIES PARAMETERIZATION OF THE PARALLEL LINEAR MODEL

T1.1 $\quad C(u,p) = p_1[h(u) \sum_{i=1}^{N} \delta_i \, p_i^* + \alpha_1 + \sum_{i=2}^{N} \alpha_i \, T^i + 1/2 \sum_{i=2}^{N} \sum_{j=2}^{N} \beta_{ij} \, T^i \, T^j]$

T1.2 $\quad \partial C/\partial u = h'(u) \sum_{i=1}^{N} \delta_i \, p_i$

T1.3 $\quad \partial^2 C/\partial u^2 = h''(u) \sum_{i=1}^{N} \delta_i \, p_i$

T1.4 $\quad \partial^2 C/\partial u \partial p_i = h'(u) \, \delta_i \qquad i = 1 \ldots N$

T1.5 $\quad \phi^i(u, p) = \partial C/\partial p_i = h(u) \, \delta_i + \alpha_i \, T_i^i + \sum_{j=2}^{N} \beta_{ij} T_i^i T^j \qquad i = 2 \ldots N$

T1.6 $\quad \phi^1(u,p) = \partial C/\partial p_1 = h(u)\delta_1 + \alpha_1 + \sum_{i=2}^{N} \alpha_i(T^i - p_i^* \, T_1^i) + 1/2 \sum_{i=2}^{N} \sum_{j=2}^{N} \beta_{ij} \, (T^i T^j - 2p_i^* \, T_1^i \, T^j)$

T1.7 $\quad s_{ii}(p) = \partial^2 C/\partial p_i^2 = p_1^{-1} \, [\alpha_i \, T_{ii}^i + \sum_{j=2}^{N} \beta_{ij} \, T_{ii}^i \, T^j + \beta_{ii}(T_i^i)^2] \qquad i = 2 \ldots N$

T1.8 $\quad s_{ij}(p) = \partial^2 C/\partial p_i \partial p_j = p_1^{-1} \, \beta_{ij} \, T_i^i \, T_j^j \qquad i,j = 2, \ldots N; \quad i \neq j$

T1.9 $\quad s_{1i}(p) = \partial^2 C/\partial p_1 \partial p_i = -p_1^{-1} \, [\alpha_i \, p_i^* \, T_i^i + \sum_{j=2}^{N} \beta_{ij}(p_i^* \, T_{ii}^i \, T^j + p_j^* \, T_i^i \, T_j^j)] \qquad i = 2 \ldots N$

T1.10 $\quad s_{11}(p) = \partial^2 C/\partial p_1^2 = p_1^{-1} [\sum_{i=2}^{N} \alpha_i \, p_i^{*2} \, T_{ii}^i + 1/2 \sum_{i=2}^{N} \sum_{j=2}^{N} \beta_{ij}(p_i^{*2} \, T_{ii}^i \, T^j + p_i^* \, p_j^* \, T_i^i \, T_j^j)]$

together with equations T1.2-T1.4 determine the δ_i and the slope and curvature parameters for h(u). Equations T1.5-T1.7 and T1.8 for $i < j$ then form a set of N(N+1)/2 linear equations for the determination of the N parameters α_i and the N(N-1)/2 independent β_{ij}. This completes the approximation since the second order derivatives with respect to the numeraire (T1.9, T1.10) are fully determined by the above parameters. There is one remaining degree of freedom that may be resolved by the choice of an initial level for the utility index h(u).

The flexibility for point approximations of the normalized Taylor series parameterization does not depend on the particular choice of numeraire. The model is thus essentially equivalent in this respect to other flexible forms, such as the Transcendental Logarithmic Cost function. In contrast with many familiar flexible forms, however, most versions of the parallel linear model exhibit global properties that are asymmetric between the numeraire and the remaining goods. An intuitive interpretation of this asymmetry is that N-1 of the Hicksian demand functions (T1.5) are essentially freely parameterized, while that for the numeraire is determined as a budget residual, $\phi^1 = \frac{1}{p_1}(C - \sum_{i=1}^{N} p_i\phi^i)$. Although the asymmetry is not itself a selling point of the model, it allows for greater than usual flexibility in parameterizing extended substitution characteristics between specific pairs of goods. This flexibility may prove particularly useful in empirical applications in which a subset of prices displays broad variation.

The comparative properties of alternative parameterizations may be illustrated by selected examples. Three straightforward normalized parameterizations are the quadratic, with $T^i(p_i^*) = p_i^*$, the generalized linear, with $T^i(p_i^*) = (p_i^*)^{1/2}$, and a modification of the transcendental logarithmic form with $T^i(p_i^*) = \ln(p_i^*)$.[6] Perpetuating a tradition of nimble nomenclature, we will refer to the latter form as the "Parallel Linear Asymetric Transcendental Semilogarithmic" or "PLATS" model, with similar acronyms, "PLGL" and "PLAQ," for the generalized linear and asymmetric quadratic models.

Cost functions for the three representative parameterizations are shown in Table 2 along with the Hicksian demand functions and substitution effects for goods other than the numeraire. For simplicity of notation and interpretation, p_1 is set to unity so that the functions are expressed only in normalized prices, p_i^*. The contrasting properties of the three parameterizations are most transparent in the forms of the cross substitution effects with respect to normalized prices. For PLAQ, a given absolute compensated change in p_j results in a constant change, S_{ij}, in consumption of good (i). Under PLATS the cross substitution effect is inversely proportional to both p_i and p_j. A given proportional change in p_j, with p_i and u constant, yields a constant absolute change in x_i, with the magnitude of the change also inversely proportional to the level of p_i. The PLGL form represents an intermediate case, with the cross substitution effects proportional to the inverse square roots of the relevant normalized prices.

The properties of the different forms are also evident from the Hicksian demand functions, ϕ^i, viewed as parametric representations of indifference surfaces. For this purpose we focus on the functions, $\psi^i(p^*) = \phi^i(u, p^*) - H(u)\delta_i$, the price sensitive components of the compensated demand functions which are independent of the level of utility in a parallel structure. Figure 2 shows representations of indifference surfaces

TABLE 2

REPRESENTATIVE FLEXIBLE PARAMETERIZATIONS OF THE PARALLEL LINEAR MODEL

A. ASYMMETRIC QUADRATIC PARAMETERIZATION; "PLAQ"

T2A.1	$C(u, p^*) = h(u) \sum_{i=1}^{N} \delta_i p_i^* + a_1 + \sum_{i=2}^{N} a_i p_i^* + 1/2 \sum_{i=2}^{N} \sum_{j=2}^{N} S_{ij} p_i^* p_j^*$	
T2A.2	$\phi^i(u, p^*) = \partial C/\partial p_i^* = h(u) \delta_i + a_i + \sum_{j=2}^{N} S_{ij} p_j^*$	i = 2....N
T2A.3	$S_{ij}(p^*) = \partial^2 C/\partial p_i^* \partial p_j^* = S_{ij}$	i,j = 2....N

B. GENERALIZED LINEAR PARAMETERIZATION; "PLGL"

T2B.1	$C(u, p^*) = h(u) \sum_{i=1}^{N} \delta_i p_i^* + \alpha_1 + \sum_{i=2}^{N} \alpha_i p_i^{*1/2} + 1/2 \sum_{i=2}^{N} \sum_{j=2}^{N} \beta_{ij} (p_i^* p_j^*)^{1/2}$	
T2B.2	$\phi^i(u, p^*) = \partial C/\partial p_i^* = h(u) \delta_i + 1/2\, \alpha_i p_i^{*-1/2} + 1/2 \sum_{j=2}^{N} \beta_{ij} p_i^{*-1/2} p_j^{*1/2}$	i = 2....N
T2B.3	$s_{ii}(p^*) = \partial^2 C/\partial p_i^{*2} = -1/4(\alpha_i p_i^{*-3/2} + \sum_{j=2, j \neq i}^{N} \beta_{ij} p_i^{*-3/2} p_j^{*1/2})$	
T2B.4	$s_{ij}(p^*) = \partial^2 C/\partial p_i^* \partial p_j^* = 1/4\, \beta_{ij} (p_i^* p_j^*)^{-1/2}$	i,j = 2....n; i ≠ j

C. ASYMMETRIC TRANSCENDENTAL SEMILOGARITHMIC PARAMETERIZATION; "PLATS"

T2C.1	$C(u, p^*) = h(u) \sum_{i=1}^{N} \delta_i p_i^* + \alpha_1 + \sum_{i=2}^{N} \alpha_i \ln p_i^* + 1/2 \sum_{i=2}^{N} \sum_{j=2}^{N} \gamma_{ij} \ln p_i^* \ln p_j^*$	
T2C.2	$\phi^i(u, p^*) = \partial C/\partial p_i^* = h(u) \delta_i + p_i^{*-1} (\alpha_i + \sum_{j=2}^{N} \gamma_{ij} \ln p_j^*)$	i = 2....N
T2C.3	$s_{ii}(p^*) = \partial^2 C/\partial p_i^{*2} = -p_i^{*-2}(\alpha_i - \gamma_{ii} + \sum_{j=2}^{N} \gamma_{ij} \ln p_j^*)$	i = 2....N
T2C.4	$s_{ij}(p^*) = \partial^2 C/\partial p_i^* \partial p_j^* = \gamma_{ij} (p_i^* p_j^*)^{-1}$	i,j = 2....N; i ≠ j

implied by each of the parameterizations for two cases of a three good model. The models are matched in their point approximation properties at the unit price equilibrium shown as the zero point of the figures. The own substitution effects at the approximation point are identical in all cases with $s_{22}(1, 1) = -3$ and $s_{33}(1, 1) = -2$. The cross substitution effect, $s_{23}(1, 1)$, has values +1 and -1 for cases 1 and 2 respectively. The figures show projections on the x_2, x_3 plane of indifference loci with p_2^* variable and p_3^* constant at 1/4, 1, and 4 (dashed lines) and similar loci with p_3^* variable and p_2^* fixed (solid lines).

Under the PLAQ parameterization, the $\psi^i(p^*)$ for $i = 2....N$ are simple linear functions of the normalized prices while $\psi^1(p^*)$ is quadratic, in keeping with the asymmetry noted earlier. The projections shown in panel A are linear, while projections in the (x_1, x_2) or (x_1, x_3) planes would be parabolic.[7] This implies satiation effects for all goods other than the numeraire. The finely dashed lines represent zero price indifference loci marking satiation levels. Larger values of the own substitution effects imply more gradual curvature of the parabolic loci and more gradual onset of satiety.

The normalized generalized linear parameterization, by contrast with the others, is not asymmetric with respect to the numeraire. The unit cost function of the PLGL form in nominal prices is in fact equivalent to the symmetric Generalized Leontief form (Diewert [12]).[8] The indifference loci between any pair of goods are hyperbolae. When all cross substitution

Substitutes Complements

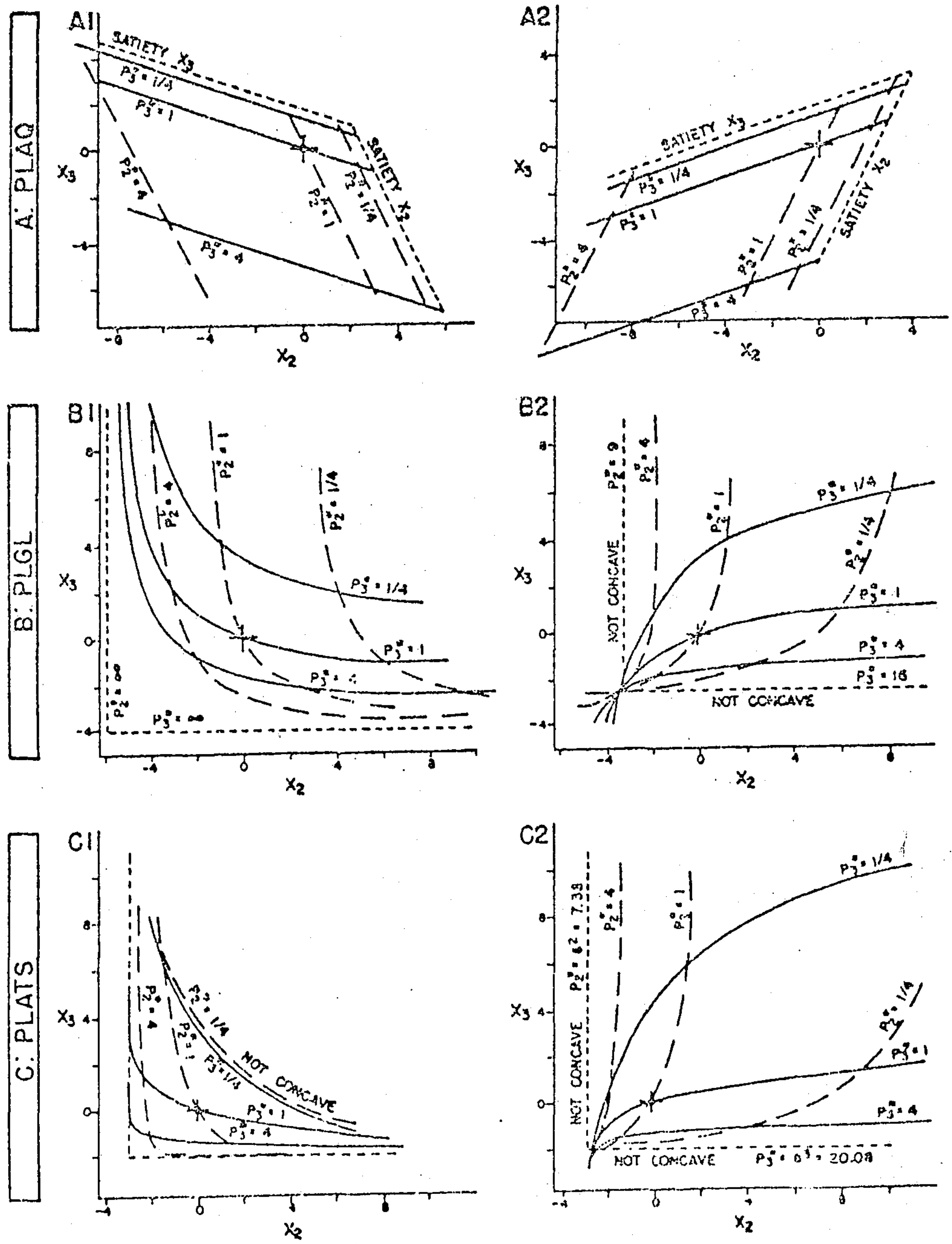

FIGURE 2: Extended Substitution Properties
Of Three Flexible Parameterizations

effects are positive, the GL structure is well behaved over a broad domain. A zero price implies infinite consumption of that good, so there are no satiation effects. On the other hand, the absolute values of substitution effects decline at high price levels so that there are asymptotic minima for consumption of all goods at any given utility level. The acceptable domain of the GL structure is more limited if any of the cross substitution effects are negative, representing complementarity of pairs of goods. The boundary of the acceptable range of the GL indifference surface is indicated by the dashed lines for $p_3^* = 16$ and $p_2^* = 9$ figure II B 2.[9]

The indifference surfaces for PLATS are asymmetric, like those for PLAQ, but are notably less simple. The inverse price proportionality in the substitution effects is stronger than the GL model and, as in that model, rules out satiation effects and implies minimum consumption levels for any given utility level. The global characteristics are also dependent on the position of the approximation point, with the α_i and the γ_{ii} jointly determining the position and the own substitution effects. The solution shown, with $\alpha_2 = 3$, $\alpha_3 = 2$, and $\gamma_{22} = \gamma_{33} = 0$ yields the simplest structure. The acceptable domain is limited, whatever the signs of cross substitution effects, with nonconcavity resulting from combinations of high prices and negative γ_{ij} or low prices and positive γ_{ij}. The boundaries shown in the figure are outer limits beyond which no combination of prices yields a concave cost function. Combinations of finite prices yield nonconcavity within the boundary as indicated in the figure by the intersection of loci with p_i^* constant at different values. These properties indicate that care is in order in the use of the PLATS parameterization.[10]

The general form of the asymmetric Taylor series expansion of the unit cost function (7) permits mixing of transformations. For example, if satiation effects are appropriate for some goods and asymptotic consumption minima for others, the $T^i(p_i^*)$ can represent linear or square root terms as appropriate. Clearly, a large number of alternative transformations could be incorporated instead. Selective additions to the Taylor series form also provide useful flexibility. For instance, the addition of linear terms in $p_i^*(\ell n p_i^* - 1)$ to the PLAQ form introduces inverse price terms in the own substitution effects and eliminates satiation effects while maintaining the simple regularity of cross substitution effects that is characteristic of that form.

With the exception of the PLGL form, the parameterizations discussed above have global properties that are asymmetric with respect to the numeraire. Such asymmetry is a disadvantage if one wishes to represent a complete and inherently symmetric demand system. In frequent applications, however, research attention is focused on a specific set of related goods, and it may be appropriate to treat all others as a Hicksian composite. In such cases the composite may be chosen as numeraire, and relative changes in its consumption will often be small enough so that the global asymmetries are inconsequential.

II. ESTIMATION FORMS FOR PARALLEL PREFERENCE STRUCTURES

Hicksian demand functions are not empirically useful in a demand theory context because utility is not directly measurable. For the case of linear parallel structures, however, conventional demand functions are easily derived by way of indirect utility functions, using Roy's Identity (Roy [21]). An indirect utility index, $V(p/y)$, is obtained by simple inversion of the cost function in $h(u)$, letting $h(u)$ be the identity function and recalling that income, y, equals expenditure, C.

$$(5) \qquad V\left(\frac{p}{y}\right) = h(u) = \frac{y - \Lambda(p)}{\Sigma\ \delta_i p_i}$$

Roy's identity then yields the conventional demand functions (6).

$$(6) \qquad x_i = \xi_i(p/y) = \frac{\partial V/\partial(p_i/y)}{\Sigma(p_j/y)\ \partial V/\partial(p_j/y)} = \frac{\delta_i[y - \Lambda(p)]}{\Sigma\delta_j p_j} + \psi^i \qquad i = 1 \ldots N$$

The demand functions (6) are nonlinear but are reasonably tractable as estimation forms. Unfortunately, demand functions of this form cannot be derived for nonlinear parallel structures because the cost function cannot be inverted in u. For such cases we instead derive a set of demand relationships in which the numeraire good provides the basis for a real income index.

The real income index is obtained by inversion of the Hicksian demand function for the numeraire good, (2, i = 1).[11]

$$(7) \qquad u = f_1^{-1}(x_1 - \psi^1(p))$$

Substitution into the remaining N-1 Hicksian demand functions then yields a set of demand relationships in terms of the observables x_1 and p.

$$(8) \qquad x_i = \zeta(x_1, p) = f_i\left(f_1^{-1}(x_1 - \psi^1(p))\right) + \psi^i(p)$$

$$= g_i(x_1 - \psi^1(p)) + \psi^i(p) \qquad i = 2 \ldots N$$

The modified Hicksian demand functions (8) have a particularly simple form in the linear case, for which the $g_i(\cdot) = f_i(f_1^{-1}(\cdot))$ are simple multiplicative factors, $g_i(u) = \frac{\delta_i}{\delta_1} u = D_i u$.

$$(9) \qquad x_i = \zeta(x_1, p) = D_i X_1 - D_i\, \psi^1(p) + \psi^i(p) \qquad\qquad i = 2 \ldots N$$

The form (9) is linear in the endogenous variable, X_1, but does involve cross products between the D_i and the substitution parameters in $\psi^1(p)$. This form is the basis for our discussion of the Ashenfelter-Heckman model in section III and will be referred to as the Generalized Ashenfelter Heckman (GAH) estimation form.

In nonlinear parallel forms the functions $g_i(\cdot)$ may be parameterized to allow for the desired curvature of income-consumption curves. In order to visualize this parameterization, it is useful to note that the argument of the g_i functions, $(x_1 - \psi^1(p))$, may itself be interpreted as a real income or utility index. The index, $U^* = x_1 - \psi^1(p)$, orders successive indifference surfaces by the level of the numeraire good at which each surface intersects the basic income-consumption curve, denoted ICC_o. The basic curve corresponds to the price vector p_o at which all the ψ^i have value zero.[12] The functions $g_i(U^*)$ for $i = 2 \ldots N$ then provide a parametric description of ICC_o and other curves are parallel, displaced by the vector $\Psi(p) = [\psi^i(p)]$.

The functions $g_i(U^*)$ may be one-parameter transformations such as $k_i \ell n(U^*)$ or $k_i \sqrt{U^*}$, or they may involve as many parameters as are necessary to allow for the desired flexibility of income effects. Polynomials in simple transformations appear to be good candidates, as do linear spline functions. A number of other forms are suggested by Lau and Tamura [19].

III. THE ASHENFELTER-HECKMAN EMPIRICAL MODEL AS A PARALLEL STRUCTURE

The empirical labor supply models estimated by Ashenfelter and Heckman ([1, 2]) specify that each spouse's labor supply, (R_i, for i=m, f), is a linear function of wage rates, w_i, and total family income, F. Family income, F, interpreted as a Hicksian composite of all market consumption, is the numeraire good in the model and the other prices, w_i, are deflated by the price of market goods. The A-H two-worker model (1974), translated from deviation form, is shown in (15).

$$(10) \quad R_i = R_o + B_i^{\dagger} F + S^{\dagger}_{im} w_m + S^{\dagger}_{if} w_f \qquad i = m,f$$

Superscripts, †, are added to the parameters $B^{\dagger}$, and $S_{ij}^{\dagger}$ to indicate that, while related, they are not generally equal to conventional income and substitution effects as they were identified to be in the Ashenfelter-Heckman analysis. As is characteristic of labor supply models, the expected signs of income and substitution parameters are reversed from those in demand models because labor is the additive complement of the good, leisure.

The labor supply relationships (15) have the same basic form as the modified Hicksian demand functions (14). The parametric form of the functions is overly astere for a multigood model, however. If a parallel structure were to have the A-H form, it would require that $\psi^{\dagger i} = (\psi^i - D_i\psi^1) = r_{io} + S^{\dagger}_{im} w_m + S^{\dagger}_{if} w_f$, where $\psi^{\dagger i}$ is defined for notational simplification below, and i = m,f. The required equalities can hold over a range of wage rates if the D_i, and hence the income effects, are equal to zero. In this special case $\psi^{\dagger i}$ equals ψ^i, and its derivatives $S^{\dagger}_{ij}$ are interpretable as substitution effects. If income effects are not zero, however, the S_{ij} are not substitution effects, and the parameters must satisfy other substantial restrictions if the interpretation in terms of parallel structures is to be maintained.

The converse of the above interpretation may be established by solving the A-H supply relations (10) to obtain conventional supply functions.

The budget constraint, $F = w_m R_m + w_f R_f + y$, is the additional equation necessary to solve for the Marshallian functions (11a, b) with arguments w_m, w_f and exogenous nonwage income, y.

$$\text{(11a)} \quad R_i(w_m, w_f, y) = \psi^{\dagger i} + \frac{B_i^{\dagger}\left[w_m\psi^{\dagger m} + w_f\psi^{\dagger f} + y\right]}{1 - B_m^{\dagger} w_m - B_f^{\dagger} w_f} \qquad i = m,f$$

$$\text{(11b)} \quad F(w_m, w_f, y) = \frac{w_m \psi^{\dagger m} + w_f\psi^{\dagger f} + y}{1 - B_m^{\dagger} w_m - B_f^{\dagger} w_f}$$

The income and substitution effects for labor supply then follow by direct differentiation of the supply function and application of the Slutsky equation.

$$\text{(12a)} \quad \partial R_i / \partial y = B_i^{\dagger}/(1 - B_m^{\dagger} w_w - B_f^{\dagger} w_f)$$

$$\text{(12b)} \quad \left. \partial R_i / \partial w_j \right|_u = s_{ij}(w_m, w_f) = S_{ij}^{\dagger} + \frac{B_i^{\dagger}(w_m S_{mj}^{\dagger} + w_f S_{fj}^{\dagger})}{1 - B_m^{\dagger} w_m - B_f^{\dagger} w_f} \qquad i = m,f \quad j = m,f$$

The substitution and income effects are, in general, functions of wage rates with the $S_{ij}^{\dagger}$ and $B_i^{\dagger}$ as parameters. If the absolute income effect is very small, as, for instance, in demand applications to a small sector of the budget, the difference between the functions and the parameters may be negligible. In the labor supply case, however, the differences are of the order of 30%. Furthermore, equality of the parameters $S_{mf}^{\dagger}$ and $S_{fm}^{\dagger}$ is not sufficient for satisfaction of the symmetry condition and global symmetry is not generally possible.[13]

The A-H parameterization can provide a good local approximation if estimated under symmetry restrictions based on the expressions (12b) evaluated at mean wage values. The local properties of the model appear to be appropriate to the Ashenfelter and Heckman empirical application to aggregate data [2], and it is reasonable to expect that the generally plausible nature of their results would be maintained under revision of the symmetry restrictions.

The A-H parameterization for one worker, $R = R_o + B^{\dagger}F + S^{\dagger}w$, is globally consistent with utility maximization.[14] The substitution effect, $s(w) = S^{\dagger}/(1 - B^{\dagger}w)$, varies slowly with the wage rate and in many applications is inconsequentially different from a constant. The A-H parameterization has the virtue of a very simple estimation form while a model with a strictly constant substitution effect (PLAQ) would entail nonlinear parameter constraints. Additional flexibility of the extended substitution properties, without sacrifice of the tractable estimation form could be gained by substitution of a transformation $T(w)$ for w in the A-H one-worker model. The substitution effect $T'(w)/(1 - B^{\dagger}w)$ would then reflect the essential characteristics of $T'(w)$.

The empirical results obtained by Ashenfelter and Heckman for their one-worker model are theoretically consistent and quite plausible. These results generally support the applicability of the parallel model, but the restrictions of the model are accepted as untested assumptions. This author [8], [10] tests these assumptions using a one-worker model that allows for nonparallel ICCs and higher order flexibility in the substitution properties. The results, for a select sample of prime-age males in labor-supply equilibrium, are supportive of the general parallel model and further suggest that the A-H parameterization of substitution properties is as good as any alternative within the range of the data. The A-H form, like the PLAQ form, implies satiety with leisure at consumption levels not far removed from the mean equilibrium. While these implications could not be rejected for a select sample from the primary labor force, general acceptance of these implications should await the outcome of more powerful tests. Overall, currently available evidence encourages further estimation and testing of the parallel preference model in labor supply applications.

IV. CONCLUDING REMARKS

In this paper we have discussed the properties of parallel preference and production structures. These structures are distinguished by indifference surfaces or isoquants that are absolutely homothetic; that is, they are the same shape and scale at all levels of utility or production. The full structures are generated by translation of these identical surfaces along parallel income consumption curves (or expansion paths) which may be either linear or nonlinear. Linear parallel forms constitute a subclass of the Gorman Polar Form and share the desirable aggregation properties of that form. Linear parallel forms also lend themselves to flexible parameterization in that they can provide a variety of second order point approximations to an arbitrary general structure. The nonlinear parallel model allows for greater flexibility over a range of income or output. The nonlinear model is directly estimable in a production context but requires the endogenous GAH estimation form in utility applications.

The absolute homotheticity of parallel structures, on the one hand, is a significantly restrictive feature, but, on the other, it underlies the particular tractability of this flexible form. The substitution characteristics and income responsiveness are specified as separable portions of the cost or expenditure function for a parallel form, and the characteristics of each may easily be tailored to the demands of a particular application. The independence of characteristics may also make parallel forms useful as a pedagogical tool. In empirical applications, flexible parallel forms are likely to be more useful in small systems where there is interest in the details of demand for interrelated goods rather than in large systems defined over broad aggregates.

The restrictions implied by absolute homotheticity appear to be quite appropriate for applications to models of individual and family labor supply. These restrictions have been supported by statistical tests in one study by this author. The two studies by Ashenfelter and Heckman [1, 2], based on approximations of the parallel model, have also yielded plausible results. A variety of applications, including testing of the multiworker model, testing for curvilinear ICCs, and the incorporation of random parameters, hold promise for future work.

A potentially useful property of the parallel model in a production context is that the PLAQ parameterization has an explicit dual in closed form (Dickinson [11]). This provides a tool for testing of the cost minimization assumptions that underlie estimation of derived demand systems. However, the absolute homotheticity of the model would limit the application to a small range of output unless the substitution effects are limited or nonexistent, as in the Lau and Tamura model [19]. As noted at the outset, such limitations are characteristic of the parallel model, but within the domain of suitable applications the model shows promise.

NOTES

[1]W. M. Gorman [14, 15]. For a recent extensive discussion, see Blackorby, Boyce, and Russell [3]. The relationship to the Gorman model, which markedly simplifies the presentation of the parallel model, was pointed out to the author by Robert Pollak and by an anonymous referee.

[2]See for instance, Diewert [13].

[3]For convenience of parameterization, the $f_i(u)$ may be negative if offset by positive linear terms in $\Lambda(p)$.

[4]Lau and Tamura [19] employed a model that may be interpreted as a nonlinear parallel model with a Leontief fixed-proportions unit cost function. I am grateful to a referee for suggesting this reference.

[5]Any flexible form can provide a point approximation to a general PLH function, but approximations that lose self-consistency away from the approximation point are of less interest in this paper given our concern with the more extended properties of the functions.

[6]Lau [18] notes that these forms provide second order approximations in a numerical sense as well as in Diewert's differential sense. In the present case, the numerical approximation applies to the unit cost function but not necessarily to the linear parallel function defined over a range of utility.

[7]Solving any pair of the ψ^i functions to eliminate the variable price yields the equation for one of the projected loci. For instance,

$$\tilde{x}_3 = \tilde{X}_3(\tilde{x}_2 | u, p_3^*) = 1/s_{22}[s_{22}a_3 - s_{23}a_2 + (s_{22}s_{33} - s_{23}^2)p_3^* + s_{23}\tilde{x}_2]$$

where $\tilde{x}_i$ is measured from $h(u)\delta_i$, gives the $(\tilde{x}_2, \tilde{x}_3)$ locus with p_2^*

variable and p_3* constant and

$$\tilde{x}_1 = \tilde{X}_1(\tilde{x}_3|u, p_3*) = (-\frac{1}{2}s_{22})[\tilde{x}_3^2 - 2a_2\tilde{x}_3 + (s_{22}s_{23} - s_{23}^2)p_3* + a_2 - 2s_{22}a_1]$$

gives the similar $(\tilde{x}_1, \tilde{x}_3)$ locus.

[8]Note that

$$p_1[\alpha_1' + \sum_{i=2}^{N} \alpha_i(p_i*)^{1/2} + 1/2 \sum_{i=2}^{N} \sum_{j=2}^{N} \beta_{ij}(p_i* \; p_j*)^{1/2}]$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} b_{ij}(p_i \; p_j)^{1/2} \qquad \text{if } \alpha_1 = b_{11};\ \alpha_i = 2b_{1i},\ \text{and } \beta_{ij} = 2b_{ij}.$$

The derivatives of the latter form are clearly symmetric.

[9]It may be confirmed that the substitution matrix is singular at those prices for which $(\alpha_2 \alpha_3 + \alpha_2 \beta_{23} p_2*^{1/2} + \alpha_3 \beta_{23} p_3*^{1/2}) = 0$. For the parameter values of figure 2B2, the pairs of finite prices (p_2*, p_3*) that imply singularity range between (9, 0) and (0, 16). All these yield identical (x_2, x_3) values shown as the intersection point at the lower left of the figure.

[10]Note that the properties illustrated are not those of the standard translog structure for which the logarithm of the cost function is represented as a Taylor expansion in logarithms. The translog unit cost function could be incorporated in the parallel model in antilog form, but the resulting expression is essentially intractable.

[11]It is assumed that f_1 is monotonically increasing; that is, that the numeraire is everywhere a normal good.

[12]Note that there is a degree of freedom in determining the levels of the ψ^i and the g_i so that p_o may be chosen to have a convenient value for a particular parameterization.

[13]The A-H two-worker model can satisfy the symmetry conditions over a

range of wage rates only if the parameters satisfy the three restrictions

$$S^{\dagger}_{mm} B^{\dagger}_{f}/B^{\dagger}_{m} = S^{\dagger}_{mf} = S^{\dagger}_{fm} = S^{\dagger}_{ff} B^{\dagger}_{m}/B^{\dagger}_{f}.$$

These restrictions are not plausible because they imply that the determinant, $(s_{mm} s_{ff} - s_{mf}^{2})$ is identically zero.

Strictly speaking, this discussion applies to a one-worker version of the A-H two-worker model. Their 1973 parameterization [1] is actually a hybrid with a transformation of a simple linear supply model because they average the imputed family income variable, F, with a standard income evaluated at a fixed level of labor supply.

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