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ABSTRACT

Lifetime income is a widely used concept in both positive and normative analysis. The definition of this concept is examined in the context of income uncertainty, capital market imperfections, and uncertainty about length of life. Two alternative, comprehensive definitions are provided and discussed. The implications for the distribution of economic well-being of various age-income profiles and differential mortality are examined using a variety of assumptions about capital and insurance markets.

The Definition of Lifetime Income

1. INTRODUCTION

"Lifetime income" has secured for itself an important role in theoretical and applied economics. The concept is obviously central to the human capital theory of earnings and the demand for schooling as well as to associated theories of occupational choice. It provides a well-known foundation for the consumption function. It enters cost benefit analysis and other studies of applied welfare economics where valuations of human lives have to be performed. It is the focus of pious hopes and practical proposals regarding improvements in the presentation and analyses of national income statistics and data on income distribution.[1]

We shall be concerned primarily with the last application, in which lifetime income is used as an index of lifetime economic well-being. This procedure is usually rationalized by considering the behavior of a rational consumer, there being a number of well-known reasons why such an index is more suitable than current income for the purposes of analyzing distributional questions. This paper examines the problem of defining and measuring such an index using the usual consumer theory approach in a world where capital market imperfections, income uncertainty, and uncertainty about one's length of life may be present. One of the basic results is that the conventional measure of lifetime income is likely to be an overestimate of the ideal index of economic well-being. We therefore ask how an ideal index can be defined, how it behaves under conditions that approximate the real world, and what the implications are for questions about income distribution.

As with many economic problems, we shall find that this problem is not essentially new. A difficulty analogous to capital market imperfections exists in atemporal multicommodity models of consumer behavior where the budget constraint is nonlinear. A commonly met practical example of this is the case where the consumption of some goods by a certain section of the population is partly subsidized by the government: one then has the problem of imputing a measure of real income to the beneficiaries of the subsidy program.[2] However, there are three essential differences. First, in the atemporal example the nonlinearity arises on account of an external distortionary influence--the government--and thus it may be legitimate to regard this interesting issue as a special case. In the models we shall examine, however, nonlinearity occurs in the very nature of the problem, and it is arguable that the "conventional" linear budget constraint is the special case. Second, the time-component of the problem means that it is irreversible and sequential, which leads one to construct the index of well-being in a specific way. Contrast this with the atemporal multicommodity case where clearly any one commodity could be the one that is subsidized, and any one could be used quite naturally as <u>numéraire</u>. Third, the atemporal example involves essentially an aggregation of expenditures that are a result of choice; the intertemporal model involves aggregation of resources--noninterest income at different stages of the life cycle.[3]

The solution to the problem of defining lifetime income is not particularly novel either, since it is closely related to the concept of "certainty-equivalence" in consumer theory under uncertainty. To see this, let us use a simple two-period model of "income-risk"[4] as a convenient introduction to some of the central issues. Suppose a person has an increasing concave utility function $U(c_0,c_1)$ where c_0

and c_1 are, respectively, "consumption today" and "consumption in the future," between which he plans to allocate all of his lifetime resources. Resources consist of an income y_0 received now with certainty, and an uncertain income, y_1, received tomorrow with a known probability distribution; there is a known, uniform rate of interest, r.

Under these circumstances, taking $\mathcal{E}U$ as a von Neumann-Morgenstern utility function, the problem becomes

$$\max_{c_0} \mathcal{E}U(c_0, [Y - c_0][1 + r]), \tag{1}$$

where $Y = y_0 + y_1/[1 + r]$, which is the conventional measure of discounted lifetime income. Assuming the existence of an interior solution to (1), we find that a necessary condition for this maximization is

$$\mathcal{E}[U_0 - [1 + r]U_1] = 0 \tag{2}$$

where subscripts denote partial derivatives. Now the consumption function derived from (2) will depend not simply on the aggregate Y, but on the particular lifetime <u>pattern</u> of noninterest income $\{y_0, y_1\}$ as long as the person is not risk-neutral. For if the person is risk-averse, increasing y_1 to $y_1 + \alpha y_1$ (where α is some small nonrandom fraction) and reducing y_0 to $y_0 - \frac{\alpha}{1 + r}\mathcal{E}y_1$ leaves expected discounted lifetime income $\mathcal{E}Y$ unaltered, but evidently increases the riskiness of the income prospect facing the individual.

What is the implication of this? In the case of pure income risk, decreasing absolute temporal risk aversion[5] is sufficient to ensure that such a pure increase in income uncertainty will increase saving. Hence we should expect a correlation between those income receivers with greater lifetime noninterest income uncertainty and those with higher levels of interest income--a result that is often typical of the self-employed. We can think of this income uncertainty in at least two ways: (a) for <u>given</u> $\mathcal{E}Y$, a higher variability of y over the lifetime (measured, say, by the observed variance of y); (b) <u>given</u> $\mathcal{E}Y$, a steeper lifetime profile of income, since this locates more of the income stream in the uncertain future. Obviously, the two interpretations overlap, but they lead us to some interesting practical conclusions. (a) If $\mathcal{E}Y$ is correlated with greater income variability (mobility) in some population, then there is a tendency for the conventional, observed distribution of income (interest and noninterest) to overstate the "true" amount of inequality. This overstatement is not just due to income mobility itself, but due to the induced effect on unearned income which is associated with the uncertainty generated by that greater income mobility. (b) If $\mathcal{E}Y$ is correlated with the "steepness" of income profiles, then once again there is an overestimate by the conventional income distribution data of the "true" amount of inequality.[6]

So far we have only discussed a possible bias in the inequality of current income--what of the inequality of lifetime aggregate income? In this case $\mathcal{E}Y$ is in itself not a suitable measure of the individual's economic position. Since the simultaneous increase in y_1 and decrease in y_0 described on page 3 will make a risk-averse person worse off, while leaving $\mathcal{E}y$ unchanged.

To see this, define $\hat{Y}$, the <u>certainty-equivalent</u> wealth, to be received with certainty in period 0, as follows:

$$\max_{c_0} \mathcal{E}U(c_0, [Y - c_0][1 + r]) = \max_{c_0} U(c_0, [\hat{Y} - c_0][1 + r]). \qquad (3)$$

Clearly $\hat{Y}$ is an accurate aggregate monetary measure of the consumer's welfare. Now suppose that the probability distribution of income prospects changes so that $\{y_0, y_1\}$ becomes riskier while $\mathcal{E}Y$ remains unaltered. Differentiating (3) and using the first-order condition (2) to simplify the expression, we obtain

$$\mathcal{E}U_1^*[1 + r]dY = U_1[1 + r]d\hat{Y}, \tag{4}$$

where dY and $d\hat{Y}$ are the resulting variations in Y and $\hat{Y}$ and the symbols * and ^ are used to denote optimal values in the maximization problems of the LHS and RHS of (3) respectively. However, given the standard definition of increasing risk in the distribution of Y in terms of second-order stochastic dominance[7] the left-hand side of (4) must be negative because of the concavity of U. Hence $d\hat{Y}$ must be negative.

Although this result is hardly surprising, it has some interesting implications. (a) Unless perfect insurance markets exist, discounted lifetime incomes will generally understate the value to all risk-averse persons of income prospects associated with particular occupations. Should these perfect markets exist, then we could take the individual income streams less the insurance premia as an accurate measure of an individual's "economic position." (b) Since such markets do not exist universally[8], the bias will obviously be greater for those whose incomes represent uninsured risks. (c) If steeper income profiles (for given $\mathcal{E}Y$) represent greater income risk, then by extension of our earlier argument, the distribution of expected lifetime income will generally overestimate the true amount of inequality. (d) The timing and type of intergenerational transfers become crucial to the pattern of "true" income inequality. Bequests may be made in human or financial form,[9] and in the absence of perfect insurance for income risks there is a motive to prefer the latter even when the rate of return on financial assets falls short of the estimated return to human capital. Furthermore, increased income uncertainty may cause the donor to advance the timing of educational bequests.

Finally, we note that uncertainty about future noninterest income is one of the primary reasons for the existence of capital market imperfections, which we discuss in the next section.

2. AN IMPERFECT CAPITAL MARKET

In this section we shall consider a simple model of a consumer's life-cycle optimization in a one-commodity certain world.[10] This will be used as a vehicle for introducing the general concept of lifetime income and to lead us to more interesting problems in the next section. Where we depart slightly from convention is that alternative possible "market conditions" will be specified. Let us deal with these first.

Consider the economic position of a person at some future age t such that $\theta \leq t \leq \bar{\theta}$ where θ is his current age and $\bar{\theta}$ is the assumed date of death. Let him have noninterest income $y(t)$, incur consumption expenditure $c(t)$, and have a current net worth $S(t)$--how these are determined will be considered later. Consider first of all a perfect capital market: all that is required at any stage is that a given interest rate $r(t)$ is used for all borrowing and lending, and that the person does not die in debt. Hence we can write market condition #1 as

$$\text{MC\#1:}\quad \begin{cases} S(\bar{\theta}) \;\geq\; 0 & (5)\\ \dot{S}(t) \;=\; r(t)S(t) + y(t) - c(t),\ \forall t\varepsilon[\theta,\bar{\theta}]\ , & (6)\end{cases}$$

where the dot denotes a time derivative. If the market conditions are made somewhat more realistic, the rate which the person borrows, $R(t)$, will be higher than that at which he lends, $r(t)$. So we get a second set of conditions:

$$\text{MC\#2:}\quad \begin{cases} (5), & \\ \dot{S}(t) \;=\; r(t)S(t) + y(t) - c(t),\ S(t) \geq 0, & (7)\\ \dot{S}(t) \;=\; R(t)S(t) + y(t) - c(t),\ S(t) \leq 0. & (8)\end{cases}$$

The ultimate extension of this, of course, is to ban all borrowing,[11] or equivalently let $R(t) = \infty$. This gives us

$$\text{MC\#3:}\quad \begin{cases} (7), & \\ S(t) \;\geq\; 0,\ \forall t\varepsilon[\theta,\bar{\theta}]\ . & (9)\end{cases}$$

Now let us consider how the variables y, c, S are determined. In this simple model we shall suppose $y(t)$ to be given as an exogenous stream $\{y(t)|t\varepsilon[\theta,\bar{\theta}]\}$.[12] Consumption is determined by choice: the person maximizes an intertemporal utility function that is increasing and concave in consumption at any age, and is additively separable:

$$W(\theta,\bar{\theta}) \;=\; \int_{\theta}^{\bar{\theta}} V(c(t),t)dt.^{13} \qquad (10)$$

Initial assets at age θ are given as S_{θ}. Hence the complete optimization problem is

$$\max_{\{c(t)\}} W(\theta,\overline{\theta}), \tag{11}$$

$$\text{S.T.} \begin{cases} S(\theta) = S_\theta, & (12) \\ \{y(t)\} = \{\overline{y}(t)\}, & (13) \\ \text{market conditions.} \end{cases}$$

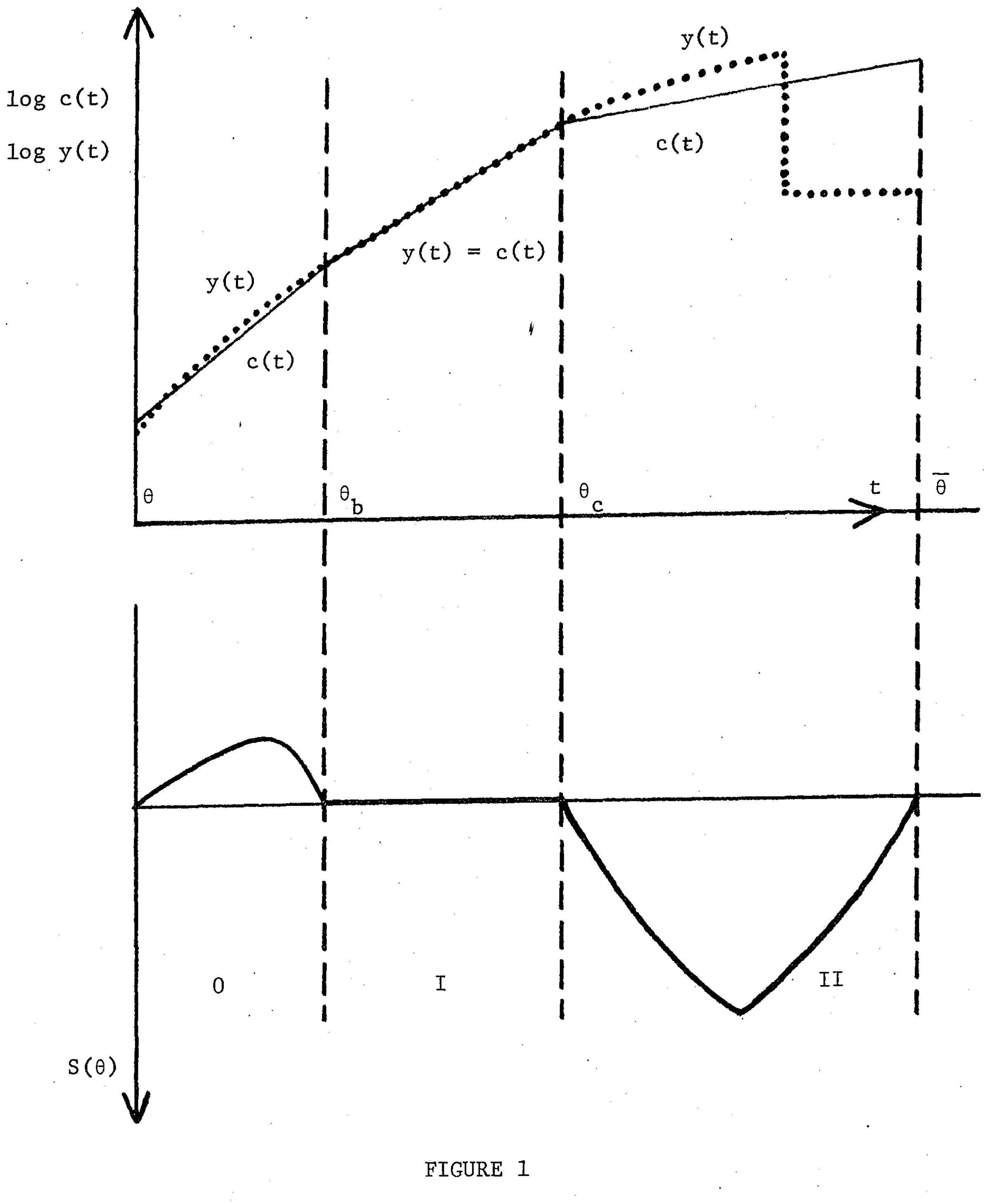
If we assume that $V(c(t);t)$ has no finite bliss point and that $\lim_{c\to 0} \partial V/\partial c = \infty$, then the solution of (11)-(13) is fairly straightforward, for any set of market conditions. Suppose MC#2 is in force, then the solution may pass through one or more of three phases, as follows:

$$\text{Phase 0:}\quad \dot{c}(t)/c(t) = [R(t) - \rho(t)]/\varepsilon(c(t)),\ S(t) < 0, \tag{14}$$

$$\text{Phase I:}\quad c(t) = y(t),\ S(t) = 0, \tag{15}$$

$$\text{Phase II:}\quad \dot{c}(t)/c(t) = [r(t) - \rho(t)]/\varepsilon(c(t)),\ S(t) > 0, \tag{16}$$

where $\rho(t) \equiv -\frac{1}{V}\frac{\partial V}{\partial t}$, the rate of pure time preference, and $\varepsilon(c)$ is $-V\frac{\partial^2 V}{\partial c^2}/\frac{\partial V}{\partial c}$, the elasticity of the instantaneous marginal utility of consumption. Transitions between phases will be determined by conditions (12) and (13), but there are obviously a number of possible stories to tell. (a) The person enters phase 0, makes a transition to phase I at age θ_b (when he finally breaks free from the loan company) and one further transition to phase II at age θ_c. This is illustrated in Figure 1, which shows possible paths of $y(t)$ and $c(t)$ in the top part, and the path of net indebtedness in the bottom part. (b) The optimal path enters phase I and stays there--income is consumed as it is acquired. (c) Entry is to phase I and there is a switch I→II at age θ_c. Here the person starts out



log c(t)
log y(t)
y(t)
c(t)
y(t) = c(t)
y(t)
c(t)
θ
θ_b
θ_c
t
$\bar{\theta}$
0
I
II
S(θ)

FIGURE 1

in life with desired consumption greater than current income, but is restricted to consuming only that income; eventually, however, $y(t)$ increases sufficiently for him to become a saver and, in later years, a dissaver out of past accumulation. This is evidently the typical story of MC#3 as well. (d) The optimal path stays in phase II. Here the profile $\{y(t)\}$ happens to be so arranged that he is able to achieve his optimum consumption profile entirely through self-financing, irrespective of the borrowing restriction. This obviously reduces to the perfect capital market case MC#1. (e) Multiple switches amongst the phases are certainly possible, but not of particular interest for us.

In what follows we shall concentrate on cases (b)-(d). This is not to deny that the other configurations, particularly case (a), are interesting, but simply to focus on a rather easily interpretable version of the model--whether or not the person suffers a welfare loss on account of capital market imperfection, i.e., whether or not the person is outside phase II. We may then look at simple propositions relating to θ_c, the phase I/phase II boundary. The neglect of phase 0 is probably not too important for three reasons. First, if $R(t)$ is sufficiently large for people with low elasticity of marginal utility, even though MC#2 holds, phase 0 will never be entered.[14] Second, ignoring the possibility that the path may be $0 \to I \to II$ rather than $I \to II$ will provide us with a lower bound on the consumer's welfare index (with MC#1 providing the upper bound). Third, the presence of phase 0 is unlikely to affect the <u>direction of movement</u> of θ_c or the other qualitative results that hold in the special cases which we shall examine in the subsequent theorems.

3. LIFETIME WELFARE REDEFINED

When we consider the question of an index of economic well-being over the person's lifetime, the obvious candidate, as we noted earlier, is the discounted sum of noninterest income, which we may write as

$$Y(\theta,\overline{\theta}) \equiv \int_{\theta}^{\overline{\theta}} e^{-\int_{\theta}^{t} r(t')dt'} \,\overline{y}(t)dt, \tag{17}$$

where r(t) is as given in MC#1. We have good reason to suspect that this is unsatisfactory for other market conditions, by simple analogy with the case of income uncertainty discussed in the introduction. If a person is unable to borrow at the rate r(t), he may actually prefer a $\{y(t)\}$ with a lower $Y(\theta,\overline{\theta})$ that happened to concentrate more of the income flow in the earlier years. We shall now make this idea more precise.

As an essential preliminary to our definitions, consider a hypothetical capital sum $\mathcal{Y}(\theta,\overline{\theta})$, which is to replace $\{\overline{y}(t)\}$ and S_θ. The optimization problem analogous to (11)-(13) is then

$$\max_{\{c(t)\}} W(\theta,\overline{\theta}) \tag{11'}$$

$$\text{S.T.} \begin{cases} S(\theta) = \mathcal{Y}(\theta,\overline{\theta}) & (12') \\ \{y(t)\} = \{0\} & (13') \\ \text{market conditions} \end{cases}$$

Let $W^*(\theta,\overline{\theta})$ denote the maximum in (11)-(13) and $W^{**}(\theta,\overline{\theta})$ the maximum in (11')-(13'). Then we can make the following two definitions:

Definition 1: $\mathcal{Y}(\theta,\bar{\theta})$ is the lifetime welfare equivalent capital sum for the income stream $\{y(t)|t\varepsilon[\theta,\theta]\}$ and initial assets $S(\theta)$ under the given market conditions if $W^{**}(\theta,\bar{\theta}) = W^*(\theta,\bar{\theta})$.

Definition 2: $y(\theta,\bar{\theta})$ is the lifetime-welfare-equivalent current income for the income stream $\{y(t)|t\varepsilon[\theta,\bar{\theta}]\}$ and initial assets $S(\theta)$ under the given market conditions if $\int_\theta^{\bar{\theta}} V(y(\theta,\bar{\theta}),t)dt = W^*(\theta,\bar{\theta})$.

The concept in Definition 1 will be called "wergild" for short, and that defined in Definition 2 the "wergild annuity."[15] Under some circumstances a simple relationship exists between the two as we shall see, but is there anything to recommend one as opposed to the other? $\mathcal{Y}(\theta,\bar{\theta})$ appears intuitively appealing since it is obviously similar to $Y(\theta,\bar{\theta})$, the usual definition. Moreover, $y(\theta,\bar{\theta})$ is subject to the charge of arbitrariness, since it is defined along the 45° ray in infinite dimensional space. However, it can be seen that under certain unusual, but possible circumstances, $\mathcal{Y}(\theta,\bar{\theta})$ may not be well defined. If $r(t)\to-1$, or if it becomes impossible to save, $S(t)$ may vanish at some point, with the result that $c(t)$ also vanishes in problem (11')-(13'). Given the assumed behavior of $V(c(t),t)$ as $c(t)\to 0$ it is clear that $\mathcal{Y}(\theta,\bar{\theta})$ would then be infinite. However, the difficulty does not exist for $y(\theta,\bar{\theta})$ as we state in the following.

Lemma 1. If $V(c(t),t)$ has a continuous, decreasing first derivative in c, then $y(\theta,\bar{\theta})$ is bounded and uniquely defined for any bounded path $\{c(t)\}$.

Proof: Let $c_0 = \min_t \{c(t)\}$, $c_1 = \max_t \{c(t)\}$ and define a function $q(x) = \int_{\underline{\theta}}^{\theta} V(x,t)dt$ on the real, positive halfline. Clearly,

$$q'(x) = \int_{\underline{\theta}}^{\overline{\theta}} V_x(x,t)dt, \tag{18}$$

and it is evident that $q'(x)$ is continuous and monotonically decreasing. Moreover, $q(c_0) \leq W(\theta,\overline{\theta}) \leq q(c_1)$, so by the continuity of $q(x)$ there must therefore exist an intermediate value $x = \mathcal{y}(\theta,\overline{\theta})$ such that $q(x) = W(\theta,\overline{\theta})$. By the strong concavity of $q(x)$, this intermediate value must be unique.

Q.E.D.

We now illustrate the use of $\mathcal{Y}(\theta,\overline{\theta})$ and $\mathcal{y}(\theta,\overline{\theta})$ by examining some of the simple cases cited in Section 2. For the explicit results we shall obtain we shall further restrict $V(c,t)$ to the form

$$V(c,t) = p(t)u(c(t)) \tag{19}$$

Obviously $-\dot{p}(t)/p(t) = \rho(t)$; if this is constant we shall say that preferences exhibit <u>intertemporal consistency</u>.[16] Also $-cu''/u' = \varepsilon(c)$; if this is constant we shall say that preferences exhibit <u>isoelasticity</u>. Furthermore, we shall find it convenient to define two functions:

$$\Omega(x) \equiv [1 - e^{x[\theta-\overline{\theta}]}]/x, \quad \text{and} \tag{20}$$

$$\Omega_c(x) \equiv [1 - e^{x[\theta_c-\overline{\theta}]}]/x, \tag{21}$$

the properties of which are discussed in Appendix A. We begin by considering the elementary case MC#1—a perfect capital market.

Theorem 1: For isoelastic preferences and a perfect capital market,

$$\mathcal{y}(\theta,\overline{\theta}) \propto \mathcal{Y}(\theta,\overline{\theta}) = Y(\theta,\overline{\theta}) + S_\theta .$$

<u>Proof</u>: If $u(c) = c^{1-\varepsilon}/[1-\varepsilon]$ and $\varepsilon > 0$, we find from (16) the optimal path to be

$$c(t) = c(\theta)e^{\int_\theta^t [r(t') - \rho(t')]dt'/\varepsilon}, \forall t \varepsilon [\theta,\overline{\theta}] \tag{22}$$

where initial consumption $c(\theta)$ is found from integration of (6), noting that (5) will hold with equality, to yield the implicit relation

$$\int_\theta^{\overline{\theta}} c(t)e^{-\int_\theta^t r(t')dt'} dt = Y(\theta,\overline{\theta}) + S_\theta . \tag{23}$$

Equations (22) and (23) together yield[17]

$$Xc(\theta) = Y(\theta,\overline{\theta}) + S_\theta, \tag{24}$$

where $X \equiv -\int_\theta^{\overline{\theta}} \exp\,(\int_\theta^t [r(t') - [r(t') - \rho(t')]/\varepsilon]dt' > 0$. Also we may use (10), (24) and definition 2 to derive, after some manipulation

$$W^*(\theta,\overline{\theta}) = \frac{1}{1-\varepsilon}\, c(\theta)^{1-\varepsilon}X = \frac{1}{1-\varepsilon}\, y(\theta,\overline{\theta})^{1-\varepsilon}\int_\theta^{\overline{\theta}} p(t)dt \tag{25}$$

Obviously from (24) and (25) $y(\theta,\overline{\theta})$ is proportional to $Y(\theta,\overline{\theta}) + S_\theta$. However, the fact that a perfect capital market $Y(\theta,\overline{\theta})$ and S_θ will appear additively in the indirect welfare function yielding $W^*(\theta,\overline{\theta})$ means that the consumer will be indifferent between the prospect of a single capital sum $Y(\theta,\overline{\theta}) = Y(\theta,\overline{\theta}) + S_\theta$ with zero subsequent noninterest income and the original prospect of a capital sum S_θ plus the income stream $\{\overline{y}(t)\}$.

Q.E.D.

Hence wergild subsumes the conventional definition of lifetime income as a special case, and as long as preferences are isoelastic and we avoid the problems mentioned on p. 12 there will be a simple relationship between wergild and the wergild annuity. Now consider case (b), cited on page 8. If the person never leaves phase I then we find

$$\mathcal{Y}(\theta,\overline{\theta}) = u^{-1}\left(\int_{\theta}^{\overline{\theta}} u(\overline{y}(t))p(t)dt \Big/ \int_{\theta}^{\overline{\theta}} p(t)dt\right) \tag{26}$$

irrespective of whether u is isoelastic. If u is isoelastic then the previous proportionality relationship between $\mathcal{Y}(\theta,\overline{\theta})$ and $\mathcal{V}(\theta,\overline{\theta})$ will hold, although, of course $\mathcal{V}(\theta,\overline{\theta})$ now no longer equals $Y(\theta,\overline{\theta}) + S_\theta$. This is illustrated in the 2-period case in Figure 2. It is interesting to compare the two polar cases where (b) the person remains throughout in phase I and where (a) the person stays in phase II. In the latter case, we deal with aggregation of (observable) income streams employing the (observable) market rate of interest. In the former case, we deal with aggregation of (subjective) utility streams employing the (subjective) rate of pure time preference. We expect to find intermediate cases involving some combination of these two approaches. Also we note that if MC#2 or MC#3 prevails, then the issue of whether lifetime welfare depends on subjective parameters will, in general, depend crucially upon the person's age.

Let us now turn to the interesting intermediate case (c)—where one transition I→II is involved, and where $\theta_c \varepsilon [\theta,\overline{\theta}]$ is the switch point between the phases. For convenience we take the case of intertemporally consistent preferences, $\overline{y}(t)$ nondecreasing in $[\theta,\theta_c]$ and a constant rate of interest r, noting that then (24) reduces to

$$c(\theta) = [Y(\theta,\overline{\theta}) + S_\theta]/\Omega(\psi), \tag{27}$$

where $\psi \equiv r - [r-\rho]/\varepsilon$ for any interval $[\theta,\overline{\theta}]$. An essential feature of the optimum is that c(t) is continuous.[18] Hence at the switch point $c(\theta_c) = y(\theta_c)$, and we may use (27) to determine the switch point θ_c by writing the corresponding relation for $[\theta_c,\overline{\theta}]$:

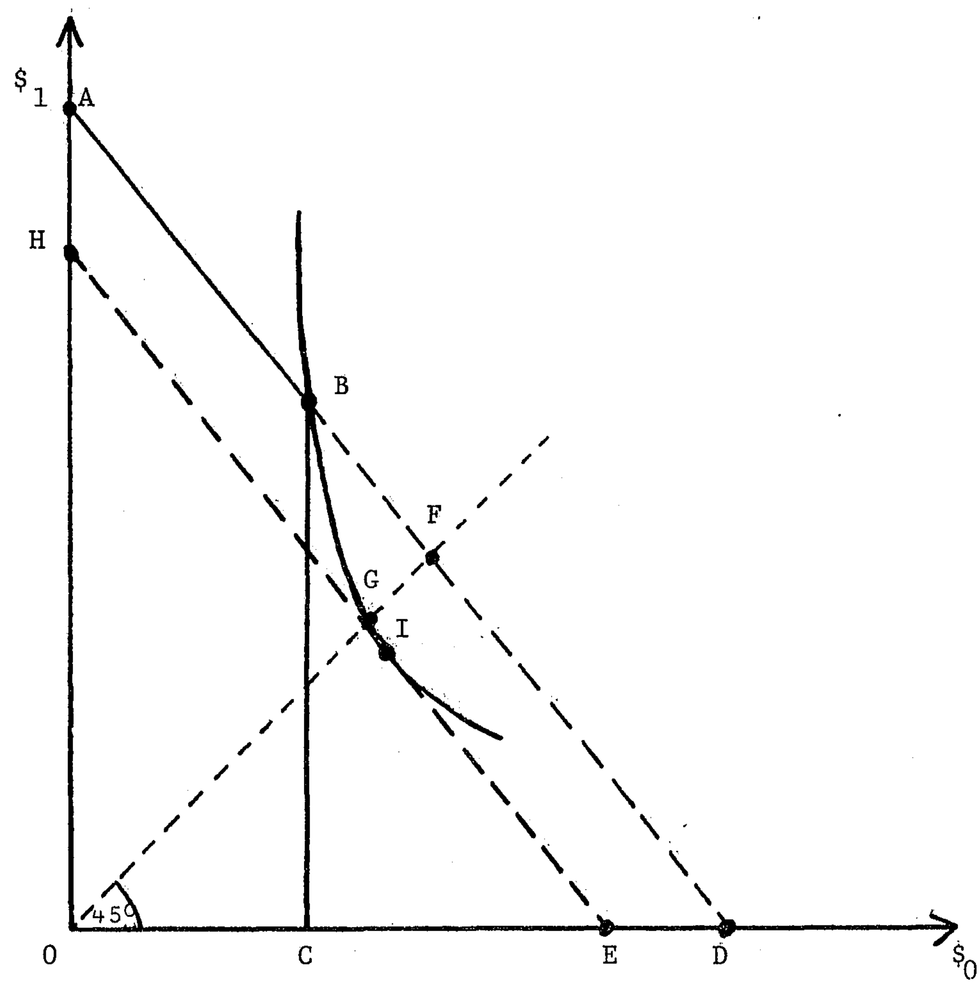


FIGURE 2

Note: Indifference curve is BGI. Actual income (equal to actual consumption) is at point B. Budget constraint is ABC. Conventional discounted lifetime income equals OD. Wergild equals OE. Weisbrod/Hansen income-net worth annuity is at F. Wergild annuity is at G. Optimum consumption would be at point I if the budget constraint were HIE.

$$Y(\theta_c,\bar{\theta}) = \bar{y}(\theta_c)\Omega_c(\psi). \tag{28}$$

So θ_c depends on the configuration of the income stream $\{\bar{y}(\theta)\}$, preferences, and the interest rate (incorporated in the parameter Ψ). Now we would expect the capital market constraint to be more "irksome" (i.e., to involve a greater loss of wergild) the more steeply sloped is the income profile. The intuitive argument is simple: the more the income profile is biased towards the early years, the less the individual would wish to have recourse to borrowing in the capital market to achieve his optimal consumption profile. Roughly speaking, then, we may take the steepness of the noninterest income profile as an indicator of the severity of the capital market constraint. Let us put this on a more formal basis.

Consider the comparison stream $\tilde{y}(t) = A + b\bar{y}(t)$, where $\tilde{Y}(\theta,\bar{\theta}) = Y(\theta,\bar{\theta})$ so that $A\Omega(r) = [1 - b]Y(\theta,\bar{\theta})$ and $db/dA = -\lambda$ where $\lambda = \Omega(r)/Y(\theta,\bar{\theta})$. The stream $\{\tilde{y}(t)\}$ is simply a displacement of $\{\bar{y}(t)\}$ that preserves its capital value. We can examine the effect of making the noninterest income profile slightly "steeper" by increasing b slightly (and reducing A) so as to preserve the value of discounted lifetime income $\tilde{Y}(\theta,\bar{\theta})$.[19]

<u>Theorem 2</u>: The steeper is the slope of the noninterest income profile under a type (b) consumption program, the greater (respectively the less) is the critical age at which the capital market constraint ceases to be binding, if optimal consumption is rising (respectively falling).

<u>Proof</u>: The condition (28) for determining the switch point gives us

$$A\Omega_c(r) + bY(\theta_c,\bar{\theta}) = [A + b\bar{y}(\theta_c)]\Omega_c(\psi). \tag{29}$$

Differentiate this w.r.t. A, and put A = 0, b = 1. Then

$$\Omega_c(r) - \lambda Y(\theta_c,\bar{\theta}) + \frac{d\theta_c}{dA}[rY(\theta_c,\bar{\theta}) - \bar{y}(\theta_c)]$$

$$= [1 - \lambda\bar{y}(\theta_c) + \frac{d\bar{y}(\theta_c)}{d\theta_c}\frac{d\theta_c}{dA}]\,\Omega_c(\psi) = [\psi\Omega_c(\psi) - 1]\bar{y}(\theta_c)\frac{d\theta_c}{dA}. \tag{30}$$

Noting equation (28), letting $\dot{c} = \frac{r-\rho}{\varepsilon}\,\bar{y}(\theta_c)$, $\dot{y} = d\bar{y}(\theta_c)/d\theta_c$ and rearranging (30) we find

$$\frac{d\theta_c}{dA} = \frac{\Omega_c(r)/\Omega_c(\psi) - 1}{\dot{y} - \dot{c}}\,\bar{y}(\theta_c). \tag{31}$$

Now $\Omega_c(x)$ is a strictly decreasing function of x for all $\theta < \bar{\theta}$. So if optimal consumption is rising, $\dot{c} > 0$ implies $r > \rho$, which implies $\psi < r$, so that the numerator of (31) is negative. By hypothesis, the denominator is positive, and so $d\theta_c/dA < 0$. Thus an increase in b (a <u>decrease</u> in A) increases θ_c. Obviously the reverse conclusion holds if $\dot{c} < 0$.

Q.E.D.

Now let us examine the effect of tilting the income profile on wergild. Since in case (c) we may take $S_\theta = 0$ we may calculate this from the equation

$$\Omega(\psi)[\mathcal{Y}(\theta,\bar{\theta})/\Omega(\psi)]^{1-\varepsilon} = [1-\varepsilon]\Omega(\rho)W(\theta,\bar{\theta}) = \int_\theta^{\theta c} e^{\rho[\theta-t]}\bar{y}(t)^{1-\varepsilon}dt$$

$$+ \bar{y}(\theta_c)^{1-\varepsilon}\, e^{\rho[\theta-\theta_c]}\Omega_c(\psi). \tag{32}$$

(Cf. Theorem 1). The sum $\mathcal{Y}(\theta,\bar{\theta})$ received at age θ would exactly compensate the individual for the loss of the noninterest income stream, taking into account the capital market constraint. To examine the effect of this constraint upon wergild for different income profiles, we may alter A, b as before.

<u>Theorem 3</u>: The more steeply sloped is the noninterest income profile under a type (b) consumption program, the lower is the value of wergild and wergild annuity.

<u>Proof</u>: Differentiating (32) and simplifying, we get

$$[\mathcal{Y}(\theta,\bar{\theta})\Omega(\psi)]^{-\varepsilon}\frac{d\mathcal{Y}(\theta,\bar{\theta})}{dA} = \left[\frac{\partial W(\theta,\bar{\theta})}{\partial A} + \frac{\partial W(\theta,\bar{\theta})}{\partial\theta_c}\frac{d\theta_c}{dA}\right]\Omega(\rho)$$

$$= \int_{\theta}^{\theta_c} e^{\rho[\theta-t]}\bar{y}(t)^{-\varepsilon}[1-\lambda\bar{y}(t)]dt + \bar{y}(\theta_c)^{-\varepsilon}\Omega_c(\psi)\, e^{\rho[\theta-\theta_c]}\left[\frac{\Omega_c(r)}{\Omega_c(\psi)} - \lambda\bar{y}(\theta_c)\right]. \tag{33}$$

Using (28) and the definition of λ we may write

$$\lambda\bar{y}(\theta_c) = \Omega(r)Y(\theta_c,\bar{\theta})/[\Omega_c(\psi)Y(\theta,\bar{\theta})],$$

which, using (17), (20), (21) we find is always less than $\Omega_c(r)/\Omega_c(\psi)$. Thus the last term in (33) is always positive. Notice that this implies that $\lambda\bar{y}(t) < 1$ for all $t \leq \theta_c$ if $y(t)$ is monotonically increasing over $[\theta,\theta_c]$. Thus the integral in (33) is nonnegative. Hence $d\mathcal{Y}(\theta,\bar{\theta})/dA > 0$ or equivalently, $\frac{d\mathcal{Y}(\theta,\bar{\theta})}{db} < 0$. Finally, notice that $y(\theta,\bar{\theta}) = [\Omega(\psi)/\Omega(\rho)]^{\frac{1}{1-\varepsilon}}\frac{\mathcal{Y}(\theta,\bar{\theta})}{\Omega(\psi)}$; so that if $\mathcal{Y}(\theta,\bar{\theta})$ is decreasing in b, so is $y(\theta,\bar{\theta})$.

Q.E.D.

Thus the intuitive argument is confirmed by the formal analysis. Notice the following interesting implications for distributional problems. (1) The age θ_c being relatively late or early is irrelevant in drawing inferences about the lifetime welfare level implicit in a particular income profile. If everyone faces the same market interest rate, then whether or not a steeper profile leads, ceteris paribus, to a longer sojourn in the constrained phase depends simply on the magnitude of the pure rate of time preference. (2) If steep profiles are correlated with high values $Y(\theta,\bar{\theta})$ for any age group θ, then inequality computed with reference to conventional discounted lifetime income will overestimate the true amount of inequality. Of course, this overestimate may be offset to some extent if more favorable borrowing/lending conditions are available to those with large lifetime incomes.

4. LIFETIME UNCERTAINTY

The extension of the basic model to uncertainty about length of life is straightforward.[20] We now assume that the person's death occurs at age T where $0 \leq T < \bar{\theta}$, T is stochastic with a known p.d.f. $q(T)$ defined on $[0,\bar{\theta}]$ and $\bar{\theta}$ may be finite or infinite. The problem (10)-(13) is modified by rewriting (12) as

$$\text{prob } \{S(T) \geq 0\} = 1, \tag{34}$$

and by writing (10) as an expected utility function

$$W(\theta,\bar{\theta}) = \int_t^{\bar{\theta}} V(c(t),t)g_\theta(t)dt \tag{35}$$

where $\forall\ t\epsilon[\theta,\bar{\theta}], g_\theta(t) \equiv \int_t^{\bar{\theta}} q(x)dx/\int_\theta^{\bar{\theta}} q(x)dx$, the probability of being alive at age t, conditional on one's having survived to age θ. For convenience, adopt a slight change of notation, writing the rate of pure time preference now as $\rho'(t)$. Then if we let $\gamma(t) \equiv -\frac{d}{dt}(\log g_\theta(t))$, the "force of mortality," we can define an <u>effective rate of time preference</u> as $\rho(t) = \rho'(t) + \gamma(t)$. On doing this we find that the differential equations (14)-(16) governing the optimal solution in its various phases hold in this modified case with the larger effective time preference rate being used. Clearly Theorem 1 still holds, and Theorems 1 and 2 will hold if the p.d.f. of T has the special property that $\gamma(t)$, the "force of mortality," is constant for all t, so that $g_\theta(t)$ has the (truncated) negative exponential form $e^{\gamma[\theta-t]}/\Omega(\gamma)$.[21]

The first task of this section is to consider the effect of differential mortality on the outcome of the consumer optimization process. The wording of this remark is quite important, since we may be interested both in changes in $g_\theta(t)$ that affect the force of mortality and <u>alter the expected life</u> $\mathcal{E}T$, and in changes that amount to a <u>pure increase in risk</u> in the conventionally defined sense.[22] In terms of the simplified survival-probability function we are interested not only in variations of the parameter γ (an increase in γ by itself reduces <u>all</u> survival probabilities and the expected survival age), but also in accompanying variations in $\bar{\theta}$ (an increase in $\bar{\theta}$ by itself leaves the force of mortality unaltered but increases the expected survival age). So, the following two theorems are of immediate interest.

<u>Theorem 4</u>: If preferences are isoelastic and there is no life insurance, an increase in the force of mortality never advances the phase switch point θ_c in a type (c) lifetime consumption program.

<u>Proof</u>: For a type (c) program a modified form of (28) must hold, namely

$$Y(\theta_c, \bar{\theta}) = X\bar{y}(\theta_c) \tag{36}$$

where $X \equiv -\int_{\theta_c}^{\bar{\theta}} \exp\left(\int_{\theta_c}^{t}[r(t') - [r(t') - \rho(t')]/\varepsilon]dt'\right)dt$. Now an increase in the force of mortality can be represented by a variation $d\rho(t)$ throughout $[\theta,\bar{\theta}]$ such that $dX \equiv -\int_{\theta_c}^{\bar{\theta}} \exp\left(\int_{\theta_c}^{t}[r(t') - [r(t') - \rho(t')]/\varepsilon]dt'\right) \int_{\theta_c}^{t}[d\rho(t')]/\varepsilon\, dt'\, dt$ is nonpositive. Differentiate (36) allowing θ_c to vary, and we find, upon simplification

$$\left[\frac{\dot{y}(\theta_c)}{\bar{y}(\theta_c)} - \frac{r(\theta_c) - \rho(\theta_c)}{\varepsilon}\right] d\theta_c = -\frac{dX}{X}. \qquad (37)$$

Now the term in brackets on the LHS of (37) is simply $\dot{y}/y - \dot{c}/c$ evaluated at θ_c, which must be positive at the phase switch point. The RHS is nonnegative. Hence $d\theta_c \geq 0$.

Q.E.D.

<u>Theorem 5</u>: An increase in $\bar{\theta}$ increases/decreases θ_c accordingly as terminal consumption in lower/higher than terminal noninterest income.

<u>Proof</u>: Once again equation (36) must hold. Differentiate this and rearrange to obtain

$$[\bar{y}(\bar{\theta}) - \bar{y}(\theta_c)e^{\int_{\theta_c}^{\bar{\theta}} [r(t) - \rho(t)]/\varepsilon dt}]e^{-\int_{\theta_c}^{\bar{\theta}} r(t)dt}\, d\bar{\theta}$$

$$= [\dot{y}(\theta_c) - \dot{c}(\theta_c)]Xd\theta_c \qquad (38)$$

Now the second term in the bracket on the LHS of (38) is simply $c(\bar{\theta})$. Since the RHS is positive, $d\theta_c/d\bar{\theta} \gtrless 0$ as $\bar{y}(\bar{\theta}) \gtrless c(\theta)$.

Q.E.D.

Let us see what we learn from Theorems 4 and 5. Increasing the force of mortality (whether for finite or for infinite maximum length of life) definitely increases the age θ_c, whatever the starting age θ. People remain longer in phase I because of the fact that the higher effective rate of time preference tilts the optimal consumption profile in favor of the present. However, unless the profile $\{y(t)\}$ has a rather peculiar shape--"turning up at the end"--rather than the more usual shape, an increase in $\bar{\theta}$ will decrease θ_c. Hence a pure increase in risk for the lifetime, requiring both $d\gamma(t) > 0$ and $d\bar{\theta} > 0$, will have an ambiguous effect on θ_c, so we do not know whether people with riskier lives spend longer in phase I. Note that Theorems 4 and 5 are true for any configuration $\gamma(t)$.

The implications of differential mortality on individual welfare raise a further difficulty. The welfare integral $W(\theta,\bar{\theta})$ includes only utility derived from consumption, not from longevity nor the "quality of life" at any stage in the life cycle. This can lead to some rather peculiar results. Consider, for example, a man who never needs to borrow in the case where $r(t)$, $\rho'(t)$, $\gamma(t)$ are constant. His lifetime welfare is given by $\mathcal{Y}(\theta,\bar{\theta}) = Y(\theta,\bar{\theta}) + S_\theta$. Now let there be a pure increase in the riskiness of the survival probabilities--in this case an increase in γ and an increase in $\bar{\theta}$ so as to keep $\mathcal{E}T$ constant. The Lorenz curve relating to the distribution of T shifts outwards[23] and the person appears to be worse off; but of course $\mathcal{Y}(\theta,\bar{\theta})$ does not decrease and will increase if the person has earning power beyond $\bar{\theta}$. This need not detain us here as long as caution is exercised when drawing inferences on the distribution of welfare in a population with differential mortality.[24]

Since consideration of a change in the force of mortality entails a change in the effective rate of time preference, let us now consider the relationship between $\mathcal{y}(\theta,\bar{\theta})$ and $\mathcal{Y}(\theta,\bar{\theta})$ when r and/or ρ change. We assume, for convenience only, r, ρ' and γ are all constant. We have for the isoelastic case

$$\mathcal{Y}(\theta,\bar{\theta}) = \Omega(\psi)^{\frac{\varepsilon}{1-\varepsilon}}\Omega(\psi)^{-\frac{\varepsilon}{1-\varepsilon}}\mathcal{Y}(\theta,\bar{\theta}) . \tag{39}$$

Differentiate (39) logarithmically:

$$\frac{d\mathcal{Y}(\theta,\bar{\theta})}{\mathcal{Y}(\theta,\bar{\theta})} = k d\rho + \frac{d\mathcal{Y}(\theta,\bar{\theta})}{\mathcal{Y}(\theta,\bar{\theta})}, \tag{40}$$

where in (40) we consider two cases:

(i) $dr = 0$, so that $d\psi = \frac{1}{\varepsilon} d\rho$ and

$$k = k_1 = \frac{1}{1-\varepsilon}\left[\frac{\Omega'(\psi)}{\Omega(\psi)} - \frac{\Omega'(\rho)}{\Omega(\rho)}\right]$$

(ii) $dr = d\rho$, so that $d\psi = d\rho$ and

$$k = k_2 = \frac{1}{1-\varepsilon}\left[\varepsilon\frac{\Omega'(\psi)}{\Omega(\psi)} - \frac{\Omega'(\rho)}{\Omega(\rho)}\right]$$

Since $\psi = r - [r-\rho]/\varepsilon$ and $\Omega'(x)/\Omega(x)$ is an increasing function of x, we see that $k_1, k_2 \gtrless 0$ as $\dot{c} \gtrless 0$. Hence the elasticity of $\mathcal{Y}(\theta,\bar{\theta})$ w.r.t. ρ is greater/less than the elasticity of $\mathcal{Y}(\theta,\bar{\theta})$ according as $\dot{c} \gtrless 0$.

If there is no life insurance, case (i) is relevant. If, further, the person never needs to borrow (case (d) in Section 2) then $\mathcal{Y}(\theta,\bar{\theta}) = Y(\theta,\bar{\theta}) + S_\theta$ which is obviously invariant with respect to ρ. Hence, in the no life insurance, no-desire-to-borrow case, wergild is independent of the force of mortality, but the annuity-equivalent rises/falls with the force of mortality as optimal consumption is falling or rising.

Let us now look at the other polar case--(b), "consume-as-you-go"--when there is no life insurance. By definition we have

$$\Omega(\rho)\mathcal{Y}(\theta,\bar{\theta})^{1-\varepsilon} = \int_\theta^{\bar{\theta}} \bar{y}(t)^{1-\varepsilon} e^{\rho[\theta-t]}dt . \tag{41}$$

Differentiate (41) with respect to ρ. We find, after some rearrangement, the following:

$$\mathcal{Y}(\theta,\bar{\theta})^{-\varepsilon} \frac{\partial \mathcal{Y}(\theta,\bar{\theta})}{\partial \rho} = \int_{\theta}^{\bar{\theta}} u(\bar{y}(t))[\theta^{*}-t]e^{\rho[\theta-t]}dt \div \Omega(\rho), \tag{42}$$

where θ^* is the mean of a random variable having the truncated negative exponential distribution with parameters ρ, $\bar{\theta}-\theta$ which will be less than $\mathcal{E}T$ as long as there is nonnegative pure time preference. Notice that the RHS of (42) will be positive/negative if $u(\bar{y}(t))$ is decreasing/increasing over $[\theta,\bar{\theta}]$. Hence $\mathcal{Y}(\theta,\bar{\theta})$ rises/falls with greater mortality as <u>actual</u> consumption is falling or rising. Hence if <u>actual</u> consumption is rising, but <u>optimal</u> consumption is constant or falling (a typical case for a high discount rate), then higher mortality implies lower wergild and lower wergild annuity. Other cases can be readily worked out.

Now consider intermediate case (c). Evidently we may write

$$\frac{du(\mathcal{Y}(\theta,\bar{\theta}))}{d\rho} = \frac{\partial}{\partial \rho}\int_{\theta}^{\theta_c} \frac{e^{\rho[\theta-t]}}{\Omega(\rho)} u(y(t))dt + \frac{\partial}{\partial \rho}\int_{\theta_c}^{\bar{\theta}} \frac{e^{\rho[\theta-t]}}{\Omega(\rho)} u(c(t))dt$$

$$+ \frac{\partial u(\mathcal{Y}(\theta,\bar{\theta}))}{\partial \theta_c} \cdot \frac{\partial \theta_c}{\partial \rho}. \tag{43}$$

The first two terms on the RHS of (43) become, after some manipulation

$$\int_{\theta}^{\bar{\theta}} u(c(t))[\theta^{*}-t]e^{\rho(\theta-t)}dt/\Omega(\rho) + \frac{1}{\varepsilon}\,\bar{y}(\theta_c)^{1-\varepsilon}e^{\rho(\theta-\theta_c)}\Omega_c'(\psi)/\Omega(\rho) \tag{44}$$

while, using equation (37) for the isoelastic case, the last term of (43) becomes $-\frac{1}{\varepsilon}\,\bar{y}(\theta_c)^{1-\varepsilon}e^{\rho(\theta-\theta_c)}\Omega_c'(\psi)/\Omega(\rho)$. Clearly then, the sign of (43) is determined by the sign of the first term in (44), and by an analogous argument to that for equation (42) it is clear that in this case too $\partial\mathcal{Y}(\theta,\bar{\theta})/\partial\rho$ will be negative if both actual and optimal consumption are rising.

We turn now to a world with life insurance. If perfect capital and insurance markets exist,[25] then we are virtually back to our original model with some of the parameters changed. If the market rate of interest on a bond in perpetuity is r', the pure rate of time preference for some person is ρ' and the force of mortality he faces is γ, then the actuarially fair rate of interest on life-insured lending/borrowing for that person is $r = r' + \gamma$, and his total personal rate of time discount is $\rho = \rho' + \gamma$ (as in the no-insurance model).[26] We may obviously apply our earlier results to this modified case. However, in order to examine the welfare effects of differential mortality we should not overlook the possibility that even if life insurance rates are actuarially fair, borrowing restrictions may still exist. Hence, we use the following, which does not require a uniform force of mortality.

Theorem 6: A higher force of mortality will lead to a later switch point θ_c under conditions of actuarially fair insurance.

Proof: Differentiate equation (36) with respect to $r(t)$, noting that actuarially fair insurance implies $d\gamma(t) = dr(t) = d\rho(t)$. For convenience write $\int_{\theta_c}^{t}[dr(t')]dt'$ as $d\Gamma(t)$ and note that $d\Gamma(t) > 0$ for at least some nontrivial interval in $[\theta_c, \bar{\theta}]$, with $dr(t) \geq 0$ everywhere. We then obtain, after simplification

$$\left[\frac{\dot{\bar{y}}(\theta_c)}{\bar{y}(\theta_c)} - \frac{r(t) - \rho(t)}{\varepsilon}\right]\bar{y}(\theta_c)Xd\theta_c = \int_{\theta_c}^{\bar{\theta}} [\bar{y}(t) - c(t)]e^{-\int_{\theta_c}^{t} r(t')dt'} d\Gamma(t)dt . \tag{45}$$

By our previous argument, the LHS of (45) is strictly positive. Now, using (7) and integrating we find that the RHS of (45) eventually simplifies to

$$\int_{\theta_c}^{\bar{\theta}} [dr(t)]S(t)e^{-\int_{\theta_c}^{t} r(t')dt'} dt, \tag{46}$$

since $S(\theta_c) = S(\overline{\theta}) = 0$. Now, by hypothesis $dr(t) \geq 0$ and $S(t) > 0$ in $(\theta_c, \overline{\theta})$, so that the RHS of (45) is strictly positive.

Q.E.D.

This enables us to examine the welfare effects of γ differing among persons when there is life insurance. The effect on $\mathcal{Y}(\theta,\overline{\theta})$ in case (b) ("consume-as-you-go") is just as it was in the no-insurance model. Now consider case (d). A higher γ implies a higher r, which in turn implies a lower $Y(\theta,\overline{\theta})$ (or $\mathcal{Y}(\theta,\overline{\theta})$). We may read off the effect on $\mathcal{Y}(\theta,\overline{\theta})$ from equation (40), where $\mathcal{Y}(\theta,\overline{\theta})$ is now given by case (ii). If optimal consumption is constant or rising, $k_2 \leq 0$ and obviously $\mathcal{Y}(\theta,\overline{\theta})$ falls with an increase in γ, although if optimal consumption is falling, the effect is ambiguous. Using Theorem 6 we find that case (c) also works out similarly to the no-insurance model, and if optimal and actual consumption are rising, we find that an increase in γ lowers the wergild annuity.

In general, then, the fact of human mortality introduces an important bias in the measurement of discounted lifetime income. In the first place, of course, were $Y(\theta,\overline{\theta}) + S_\theta$ to be used, the correct discount rate to use is $r = r' + \gamma$, <u>not</u> r', and lifetime incomes computed from the observed incomes of survivors must be adjusted for mortality accordingly. Second, $Y(\theta,\overline{\theta}) + S_\theta$ provides an upper bound to the true measure $\mathcal{Y}(\theta,\overline{\theta})$. Third, if actual and optimal consumption profiles are rising, then $\mathcal{Y}(\theta,\overline{\theta})$ is lower in models with mortality than those without.

5. CONCLUDING REMARKS

Let us consider the implications of the foregoing for income distribution analysis. Take a subset of the population the members of which are all of θ years of age, have apparently identical resources (the same value of S_θ and the same value of $Y(\theta,\overline{\theta})$ evaluated using r', the risk-free rate of interest), and identical tastes, and who all face market conditions 2 or 3. We realize that some suffer significant welfare losses because the capital market is not perfect. Which are the most disadvantaged? We know that this depends on the person's income slope and survival probability. Intuition might prompt us to accept each of the following four propositions.[27]

A. Those of the group with steeper $y(t)$ profiles have lower $\mathcal{Y}(\theta,\overline{\theta})$.

B. Those of the group with riskier lives have lower $\mathcal{Y}(\theta,\overline{\theta})$.

C. Those of the group with greater mortality have lower $\mathcal{Y}(\theta,\overline{\theta})$.

D. Those of the group with a later θ_c have lower $\mathcal{Y}(\theta,\overline{\theta})$.

However, only A is generally true.[28] The fact that D is false is important empirically. If θ_c is observable it might appear that the length of the constrained period $\theta_c-\theta$ provides a simple indication of which members of the group of θ-year-olds suffer a greater reduction in real income. Unless further information is available, however, this apparently attractive link is illusory--even among persons with identical tastes and aggregate resources. The information that we need, of course, is the slope of actual and optimal consumption profiles: if these happen to be positive, then C and D would be true. Otherwise, those with steeper profiles will have lower wergild annuity (but not necessarily a later θ_c); and those with greater mortality will have later θ_c (but not necessarily lower wergild annuity).

Let us suppose A, C, and D are true in this population. What are the implications for the entire group of age θ? First, note that if a high values of $Y(\theta,\overline{\theta})$ correlate with steep $\{y(t)\}$ profiles,[29] then, as we have argued in the case of income uncertainty, inequality in terms of conventionally measured discounted lifetime income will *ceteris paribus* be biased upwards from the true value of inequality. However, this effect will be offset to the extent that S_θ (inherited wealth) is correlated with $Y(\theta,\overline{\theta})$. So to provide a definitive answer about the likely bias on inequality amongst θ-year-olds we need to know the *joint* distribution of S_θ, $Y(\theta,\overline{\theta})$ and the slope of $\{\overline{y}(t)\}$. Second, if high ρ is correlated with low resources (either because mortality or pure time preference is high among the poor), then the distribution of $Y(\theta,\overline{\theta}) + S_\theta$ will tend to *underestimate* the inequality of real income, whatever the market conditions. Third, if the market condition one faces depends on one's assets, so that poor people face more penal borrowing rates and lower overdraft ceilings, then this underestimate will be further accentuated.

When we assemble the different age groups in the population we catch people in different phases of their life-cycle program; one of age θ_0 may still be in phase I, while another of identical characteristics but of age $\theta_1 > \theta_0$ may have passed into phase II. The former will generally have $\mathcal{Y}(\theta,\overline{\theta}) < Y(\theta,\overline{\theta}) + S_\theta$ while the latter has $\mathcal{Y}(\theta,\overline{\theta}) = Y(\theta,\overline{\theta}) + S_\theta$. If people make at most one transition from phase I to phase II then, *ceteris paribus*, the proportion of θ-year-olds for which $Y(\theta,\overline{\theta}) + S_\theta$ is an overestimate will decrease with θ. Unless there is a significant proportion of the population which remains in phase I throughout the life cycle, we thus expect measured inequality for the θ-year-olds using wergild to converge to measured inequality in terms of conventional discounted lifetime income. We noted earlier a bias between observed and real inequality in any one group. Now if inequality in the total population is some weighted sum of the inequality in each age group with the weights depending on the age structure of the population, then the overall bias obviously depends on the age distribution. In particular, we expect the bias to be larger the greater is the birth rate and the greater is the force of mortality. Interestingly, then, we find that survival probabilities enter into the divergence between "conventional" and "true" lifetime income inequality in two ways--once through the determination of the lifetime income concept, and once in the aggregation procedure.

APPENDIX A

The Truncated Exponential Distribution

The distribution under consideration has two parameters, γ, $\bar{x}$, each of which is assumed to be strictly positive. The density function is

$$f(x) = \frac{\gamma}{\Omega(\gamma)} e^{-\gamma x}, \quad x\varepsilon[0,\bar{x}]$$

$$= 0 \text{ elsewhere}$$

where $\Omega(\gamma)$ is the function $\frac{1}{\gamma}[1 - e^{-\gamma\bar{x}}]$, as in the text. Evidently we have the distribution function

$$F(x) = [1 - e^{-\gamma x}]/\gamma\Omega(\gamma) .$$

Also we may derive

$$\int_0^x t\, f(t)dt = [1 - e^{-\gamma x} - \gamma x e^{-\gamma x}]/\gamma^2\Omega(\gamma) \tag{A1}$$

From (A1) we may read off $\mathcal{E}x$, by setting $x = \bar{x}$, and also the first-moment function $F_1(x)$, which is simply (A1) divided by $\mathcal{E}x = -\Omega'(\gamma)/\Omega(\gamma)$. Define a transformed parameter $\alpha = \gamma\Omega(\gamma)$. Then, on substitution, it can be found from (A1) that the first moment function is

$$F_1(x) = \frac{\alpha F(x) + [1 - \alpha F(x)] \log(1 - \alpha F(x))}{\alpha + [1 - \alpha] \log(1 - \alpha)} \tag{A2}$$

We may use (A2) directly to give us the equation of the Lorenz curve which in this case is

$$F_1 = \chi(\alpha F)/\chi(\alpha),$$

where $\chi(u) = u + [1 - u] \log(1 - u)$, $u\varepsilon[0,1]$. It may easily be checked that $\partial F_1/\partial F$ and $\partial^2 F_1/\partial F^2$ are strictly positive for α, $F\varepsilon(0,1)$. To examine the family relationship of the Lorenz curves, vary α. We find

$$\frac{\alpha}{F_1}\frac{\partial F_1}{\partial \alpha} = \alpha F\,\frac{\chi'(\alpha F)}{\chi(\alpha F)} - \alpha\,\frac{\chi'(\alpha)}{\chi(\alpha)}\,. \tag{A3}$$

Obviously (A3) will be positive or negative as the elasticity function $\eta(u) \equiv u\chi'(u)/\chi(u)$ decreases or increases in u. Differentiation of this elasticity function gives us, after simplification

$$\frac{\partial\eta}{\partial u} = [u^2/[1 - u] - [\log(1 - u)]^2]/[+] \tag{A4}$$

A series expansion of the first and second terms of the numerator reveals that (A4) is strictly positive for all $u\varepsilon(0,1)$. Hence, η increases in u over the interior of the unit interval, and thus, from (A3), $\partial F_1/\partial\alpha$ must be negative. Since this implies that $\partial F_1/\partial\gamma < 0$, we can see that successively higher values of γ shift the Lorenz curve "monotonically outwards." Finally, note that in the special case $\overline{x} = \infty$, $\mathcal{E}x$ becomes $1/\gamma$ and the family of Lorenz curves degenerates to a single member $F_1 = \chi(F)$.

APPENDIX B

The Existence of "Phase 0"

We assume $R(t)$, $\rho(t)$, ε constant, and write $\psi = R-[R-\rho]/\varepsilon$. Since the optimal plan is continuous we must have $c(\theta_b) = y(\theta_b)$ where θ_b is the boundary between phases I and II. Hence integrating income and consumption over $[0,\theta_b]$ and noting that $S(0) = S(\theta_b) = 0$, we get, after simplification,

$$y(\theta_b)\,\frac{1}{\psi}\,[e^{\psi\theta_b} - 1] \;=\; Y(0,\theta_b)\;. \tag{B1}$$

Writing $Z \equiv Y(0,\theta_b)/Y(\theta_b)$, (B1) yields

$$\theta_b \;=\; \log(1 + Z\psi)/\psi. \tag{B2}$$

If θ_b exists, then it must be positive. However, if $\psi \le -\frac{1}{Z}$, θ_b cannot exist, and neither can phase 0. If $\varepsilon < 1$ this occurs when

$$R \;>\; [\tfrac{\varepsilon}{Z} + \rho]/[1 - \varepsilon]\;.$$

NOTES

[1]See, for example, Weisbrod (1962), Weisbrod and Hansen (1968), Stoikov (1971), Royal Commission on the Distribution of Income and Wealth (1976), Layard (1977), von Weizsäcker (1976). Note also the interest in inferring the distribution of lifetime income from data on the distribution of current income--e.g., the comments by Johnson (1977), Kurien (1977) on the work of Paglin (1975).

[2]See, for example, the work of Smolensky et al. (1977) and Smeeding (1977).

[3]Clearly noninterest income may, in practice, be endogenous to the consumer optimization problem as a result, say, of human investment. This in fact raises some serious conceptual problems for the definition of a measure of lifetime economic well-being which are beyond the scope of this paper.

[4]See Dreze and Modigliani (1972).

[5]In other words, $-U_{11}/U_1$ decreases with an increase in c_1 offset by a decrease in c_0. See Sandmo (1970).

[6]This conclusion itself needs to be modified for two reasons. First, although $\mathcal{E}Y$ may be correlated with steep income profiles, very low values of $\mathcal{E}Y$ are also likely to be correlated with high income variability of type (a) because of sickness and unemployment. Second, the analysis is only true for "income risk," and does not necessarily apply to "capital risk," where the rate of interest is a random variable. However, the subsequent discussion of "certainty equivalence" is also applicable to capital risk.

[7]See Hadar and Russell (1969) for a definition. A further problem which we shall not explore here is that the reduction in $\hat{Y}$ may depend on the age-viewpoint of the individual, since it is conceivable that one's intrinsic risk aversion alters with age.

[8]Thus, while market arrangements, or social provision, may cover certain potential losses of income due to sickness, disability, or involuntary unemployment, for a number of obvious reasons, insurance contracts or state benefits will not be found covering "income losses" arising from "not getting promotion" or "not passing one's exams." Yet the implicit income variability involved here is extremely important for the person's lifetime decision-making--see Levhari and Weiss (1974), who discuss among other things the constraints on the person's consumption possibilities resulting from income uncertainty.

[9]See Blinder (1976), Ishikawa (1974, 1975).

[10]The basic model is, of course, well known from Yaari (1964) and many other references.

[11]This can easily be adapted to a nonzero ceiling on borrowing; the reason we have not pursued this further is the needlessly complicated pattern of phase-switching which one then must take into account. Capital market conditions of type MC#2 and MC#3 have also been considered by Flemming (1973) and Pissarides (1978).

[12]Where there is no danger of ambiguity, this expression will be written $\{y(t)\}$.

[13]A bequest motive could easily be incorporated but is an unnecessary complication.

[14]See Appendix B for a proof of this.

[15]$\mathcal{Y}(\theta,\bar{\theta})$ is a generalization of "lifetime wealth" used in Pissarides (1978) and is analogous to the "equivalent income" definition of Smolensky et al. (1977) in this atemporal analysis. It is evident from our subsequent analysis that $\mathcal{Y}(\theta,\bar{\theta}) = Y(\theta,\bar{\theta}) - S(\theta)$ is the compensating variation required for giving the person access to a perfect capital market. The quantity $y(\theta,\bar{\theta})$ is a generalization of the "utility equivalent annuity income" used by Nordhaus (1973). "Wergild" has an interesting history. In ancient Teutonic and Old English law, it was the price set upon a man according to rank, paid by way of compensation or fine in the case of homicide and certain other crimes. See also Creedy (1977).

[16]See Strotz (1956).

[17]In fact we can easily show that

$$c(t) \;=\; \phi(Y(\theta,\bar{\theta}) \;+\; S_{\theta},\, t)$$

regardless of whether preferences are isoelastic.

[18]This follows because $V(c(t), t)$ has a first derivative in $c(t)$ that is continuous and monotonically decreasing. See Yaari (1964).

[19]This is analogous to the procedure adopted by Sandmo (1970) for examining a special case of pure increase in risk.

[20]See Yaari (1965).

[21]This implies that expected date of death conditional on survival to θ is $T = \theta - \Omega'(\gamma)/\Omega(\gamma)$, where $\Omega'(\gamma) < 0$, $\Omega''(\gamma) > 0$, $\partial\mathcal{E}T/\partial\gamma \leq 0$, $\partial\mathcal{E}T/\partial\bar{\theta} \geq 0$. Even if we do not adopt this assumption, the main results of Theorems 2 and 3 still hold, but the statement of them must be modified.

Also, for the purposes of this paper, we shall assume that survival probabilities are independent of past or current income or wealth--thus removing a further possible "endogeneity" problem in the form of the purchase of health-care services as part of the life-cycle optimization program.

[22]The importance of this distinction has been brought out by Levhari and Mirman (1977).

[23]For proof of this, see Appendix A. A similar problem concerning welfare has been noted by Katz (1979).

[24]The fact that we exclude intrinsic benefits from longevity means that the benefits from life insurance which we examine below do not represent the full effects. For this reason also (among others), the insured value of a person's life will not in general equal lifetime income discounted at an actuarially fair rate. This distinction has been noted by Mishan (1971) in his four methods of computing the value of a person's life--corresponding roughly to our concept of Wergild.

[25]In Yaari's (1965) terms this means perfect markets in both "regular notes" and "actuarial notes." Thus the equation for the rate of growth of optimal consumption is exactly the same as for the no-mortality case--equation (6).

[26]Actuarially fair life insurance eliminates the effect on this rate of growth introduced by the higher effective time preference rate. See Yaari (1965) and Barro and Friedman (1977).

[27]We beg an important question in that we assume that it is possible to observe θ_c and distinguish those of the group for whom $c(t) = y(t)$ at some moment happens to be optimal, and those for whom $c(t) = y(t)$ is imposed by imperfections in the capital market. Obviously if we do not

or cannot make this distinction we may be led astray to a considerable extent--see, for example, the criticisms that have been made of Thurow (1969) in this regard.

[28]See Theorem 3.

[29]If noninterest income is endogenous, however, there is a further problem, since a particular occupation and associated $\{\overline{y}(t)\}$ stream will be chosen with S_θ and the relevant market condition in mind. Under such circumstances human investment decisions will depend on financial wealth S_θ, and on the slope of $\{y(t)\}$, not just the aggregate $Y(\theta,\overline{\theta})$.

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