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SOME IMPLICATIONS OF UNCERTAINTY FOR FIRM AND INDIVIDUAL BEHAVIOUR UNDER A NEGATIVE INCOME TAX PLAN

Various hypotheses have been established concerning the effect on firm and individual behaviour of the imposition of a negative income tax plan. Many of these hypotheses are drawn directly from economic theory under certainty. The purpose of this paper is to illustrate certain conclusions which can be derived from economic theory under uncertainty and to point out whether or not they differ from their counterparts obtained under certainty.

The effect of a negative income tax on the "riskiness" of an income stream is considered in Part I. If it can be established whether or not a negative income tax system reduces the risk associated with a particular income stream, it may be possible to infer behavioural responses to such a scheme directly from the literature on uncertainty. This section is illustrative in that no general hypotheses are established. In Part II the reaction of the self-employed to the imposition of a negative income tax is explored; where the self-employed are defined to be individuals who manage their firms.

I. Income and Risk Under a Negative Income Tax Scheme

Under a negative income tax scheme an individual's income (I^N) is given by:

$$I^N = \begin{cases} G + (1-t)\, I & \text{for} \quad 0 \leq I \leq G/t \\ G + I & \text{for} \quad I < 0 \\ I & \text{for} \quad I > G/t \end{cases} \qquad (1)$$

where $G \geq 0$ is the guarantee,

$1 > t > 0$ is the tax rate,

I is the individual's income in the absence of a negative income tax.

Unless stated otherwise it is assumed that $0 \leq I \leq G/t$.

Assume that there exists some (subjective) probability distribution over I which is unaltered by the imposition of a negative income tax; then the expected values of income without (E(I)), and with a negative income tax plan ($E(I^N)$) are:

$$E(I) = \int_0^{G/t} I\, f(I)\, dI = \bar{I}$$

$$E(I^N) = G + (1-t)E(I) = G + (1-t)\bar{I}$$

and the difference between the two means is given by

$$E(I^N) - E(I) = G - t\bar{I} \geq 0 \quad \text{for } 0 \leq I \leq G/t. \tag{2}$$

The measures of risk proposed by Rothschild and Stiglitz (1970) can be used to compare the riskiness of different income streams. They have demonstrated that the following measures of risk are equivalent:

(i) If X and Y are two random variables and Y is equal to X plus another random variable Z, where $E(Z|X) = 0$ for all X, then Y is more risky than X.

(ii) If X and Y are two random variables with the same means, then Y is more risky than X if X is preferred to Y by a risk-averse individual. Assuming the existence of a Von Neumann-Morgenstern utility function, $U(\cdot)$, Y is more risky than X if $E[U(X)] > E[U(Y)]$ and the utility function is concave.[1]

(iii) If X and Y are two random variables with the same means, then Y is more risky than X if the distribution of Y has heavier tails than the distribution of X.

Following the definition of riskiness in (ii), I^N is less risky than I if the means of these two distributions are identical and if a risk-averse individual prefers I^N to I. At issue is whether or not a risk-averse individual would choose to operate under a negative income tax scheme even if it did not alter his expected income. Or, in terms of expected utility, and correcting for the difference in means $E(I^N) - E(I)$, is

$$E[U(G + (1-t)I - [E(I^N)-E(I)])] \gtrless E[U(I)],$$

assuming that the utility function is concave? It follows from Equation (2) that

$$\begin{aligned} E[U(G + (1-t)I - [E(I^N)-E(I)])] &= E[U(G + (1-t)I - [G-t\bar{I}])] \\ &= E[U((1-t)I + t\bar{I})] \quad . \end{aligned}$$

Since the utility function is strictly concave and $0<t<1$

$$U((1-t)I + t\bar{I}) > (1-t)U(I) + tU(\bar{I}) \text{ for all } I,$$

and consequently

$$\begin{aligned} E[U((1-t)I + t\bar{I})] &> E[U(I)](1-t) + t\,U(\bar{I}) \\ &> E[U(I)] + t[U(\bar{I}) - E[U(I)]] \quad . \end{aligned} \tag{3}$$

The utility function, being strictly concave, implies by Jensen's inequality that

$$E[U(I)] < U(E(I)) = U(\bar{I}),$$

and hence

$$t[U(\bar{I}) - E[U(I)]] > 0. \tag{4}$$

Equations (4) and (3) imply that

$$E[U(G + (1-t)I - [E(I^N) - E(I)])] > E[U(I)] \tag{5}$$

for all $0 \leq I \leq G/t$. The result stated in Equation (5) means that an individual who is averse to risk would prefer the negative income tax plan even if his expected income under this plan was the same as it would be without the plan. From definition (ii) above, this result suggests that the imposition of a negative income tax scheme will reduce the risk associated with a particular income stream.

Although this definition of riskiness is widely used, specific conclusions which have been derived in the literature about behavioural responses to more or less risky income cannot be employed directly to draw inferences concerning the effect of a negative income tax, for two reasons. First, the assumption that the distribution of income is unaffected by the imposition of a negative income tax is generally untenable. Economic theory suggests that an individual's earned income will change as he or she reacts to a negative income tax plan. However, it may be valid to assume that the income streams of certain subgroups of the population, retired persons, for example, would be unaffected by the adoption of a negative income tax scheme. Second, even if individual's income streams are not altered, the expected value of income plus payments differs from that of earned income, and this must be taken into consideration when attempting to draw inferences from the uncertainty literature. For these reasons, a more fruitful approach to deriving hypotheses about responses to the introduction of a negative income tax plan, is to evaluate explicitly the introduction of the guarantee and tax rate.

II. Adjustments to a Negative Income Tax Plan by the Self-Employed

The effect of a negative income tax on firm and individual decisions is explored by first determining the conditions which describe an optimum and then ascertaining the effect of changing the tax rate and the guarantee. It is assumed that an individual's earned income is generated by a firm controlled by him, and in some instances that it is supplemented by wage income.

Income: The Sole Objective

In this section the individual is assumed to manage but not work in the firm he operates, and consequently to seek to maximize expected utility of income only. Also he is assumed to be averse to risk.

Price and output uncertainty. A model in which a firm operator chooses the level of output with certainty but faces an uncertain demand expressed as a (subjective) probability distribution over price is described in detail in Appendix A. The individual's attitude towards risk is summarized by a Von Neumann-Morgenstern utility function, $U(\cdot)$, and he wishes to choose the level of output that maximizes his expected utility of income. More formally, the firm operator's objective is assumed to be

$$\max_{I} E[U(I)] ,$$

subject to $I = G + (1-t)\pi$,

where $\pi = PX - C(X) - B$ is profit, X is output, P is output price, $C(X)$ is the variable cost function and B is fixed costs. The equilibrium condition defining the optimal level of output is

$$E[U'(I)\ (P- C'(X))] = 0.$$ [2] (6)

The effect on output of an increase in the guarantee is given by

$$\frac{\partial X}{\partial G} \doteq - \frac{1}{D} E[U''(I)(1-t)(P-C'(X))] , \qquad (7)$$

where $D = E[U''(I)((1-t)(P-C'(X)))^2 - U'(I)(1-t)C''(X)]$.

It is demonstrated in Appendix A that the expression in (7) is positive if it can be assumed that the individual is more willing to enter small gambles of fixed amounts as his income increases. More formally, output will increase as the guarantee increases if decreasing absolute risk aversion can be assumed. The effect on output of an increase in the tax rate is given by

$$\frac{\partial X}{\partial t} = \frac{(1-t)}{D} E[U''(I)\pi(P-C'(X))] ,$$

which is positive if relative risk aversion is increasing, otherwise it is indeterminate. Increasing relative risk aversion is implied by the hypothesis that if both wealth and the size of the bet are increased in the same proportion, the willingness to accept the bet should decrease. The effect on output of changing the tax rate or the guarantee can be used to infer the effect of these changes on the quantities of variable inputs employed. If, for example, output expands as a result of increasing the guarantee, the use of normal variable inputs will increase whereas the use of regressive factors will decrease.

A model in which a firm owner wishes to maximize his expected utility of income, where the demand is known with certainty but the level of output is uncertain, is described in Appendix A, Note (2). It is a short-run model wherein capital is fixed ($\bar{K}$) and the individual chooses an amount of labour (L) at a particular wage rate in order to maximize his expected utility of income. Once the optimal amount of labour has been chosen, there exists some (subjective) probability distribution over output. The firm owner's objective is to

$$\max_{I} E[U(I)] \ ,$$

subject to $I = G + (1-t)\pi$,

where $\pi = PF(\bar{K},L) - wL - B$ is profit, w is the wage rate and B is fixed costs. The consequences of an increase in the guarantee and the tax rate are similar to that established above. An increase in the guarantee will increase the amount of labour used in the firm if decreasing absolute risk aversion is assumed. The amount of labour used in the firm will increase if relative risk aversion is increasing and the tax rate increases.

While decreasing absolute risk aversion is an intuitively reasonable behavioural hypothesis which has not been refuted in the literature, controversy surrounds the assumption of increasing relative risk aversion which has been propounded by Arrow (1971). If it can be accepted that the operator of a firm is simply interested in maximizing expected utility of income, the short-run effect of an increase in the guarantee level is likely to be an increase in output supplied and labour used by the firm. No such conclusion can be drawn concerning the effect of an increase in the tax rate.

Inspection of the individual's assumed criterion function reveals the symmetry between fixed costs and the guarantee level. Sandmo (1971) has pointed out that where an individual is maximizing his expected utility of income, where the income is generated by firm profits, an increase in fixed costs will reduce output. The reason for this and the effect of the guarantee in the model outlined above follows from the result that under uncertainty, risk-averse firms produce less than risk-indifferent firms.[3] If the firm has a decreasing (increasing) absolute risk aversion function then output should increase (decrease) with increasing income. Increasing the guarantee or decreasing fixed costs has the same effect as increasing income.

Price and output certainty. In the absence of uncertainty and where the individual's objective is assumed to be to attain the highest utility of income, changing the guarantee or the tax rate will not affect either the level of output or the amount of labour used in the firm. Returning to the situation in which the firm owner chooses output to maximize expected utility of income, Equation (6) implies that under certainty, price equals marginal cost at the optimum. Hence $\partial X/\partial G = \partial X/\partial t = 0$ from Equations (7) and (8). In Appendix A, Note (2), where the model in which the firm chooses an amount of labour to maximize expected utility of income is described, Equation (A12) defines the marginal value product of labour used in the firm to equal the wage rate at the optimum under certainty. It follows that the optimal amount of labour used is not altered by changing the guarantee or the tax rate.[4]

The result that under certainty neither the tax rate nor the guarantee affects firm decisions stems from the specification of the firm owner's objective. Maximizing utility of income is equivalent to maximizing profits since the former is simply a monotonic transformation of profits.

When a firm manager works in his firm his labour is not in perfectly elastic supply at a particular wage rate; and hence it is necessary to incorporate the firm operator's consumption-leisure trade off in his objective.

An Evaluation of the Firm Operator's Choice Between Income and Leisure

It is assumed that the individual is averse to risk and that his preferences for consumption and leisure can be summarized by a utility function defined over income and leisure. This section is divided into two parts. The first part is restricted to those firm operators who work in their firms and for wages; the second part concerns those firm operators who work in their firms only. It is assumed throughout this section that individuals do not respond to changes in the tax rate and the guarantee by altering their decision to participate in the wage market.

Work in the firm and for wages. The model described in Appendix B assumes that the firm owner chooses the amount of his labour supplied to the firm and the time spent working for wages which maximizes his expected utility of income and leisure. Specifically it is to

$$\underset{I\ L_1\ L_2\ L_3}{\text{maximize}} \quad E[U(I, \bar{L}-L_1-L_2-L_3)] \ ,$$

subject to $I = G + (1-t)y$,

where $y = P_1F^1(\bar{K}_1,L_1) + P_2F^2(\bar{K}_2,L_2) + wL_3 - B$ is earned income,

L_i is the amount of labour supplied in the production of the ith output, $Q_i = F^i(K_i,L_i)$ $i=1,2$,

p_i is the price of the ith output,

L_3 is the hours worked outside the firm for the certain wage rate w,

the capital inputs, K_1 and K_2, are fixed at $\bar{K}_1$, and $\bar{K}_2$ respectively.

Although P_2 is known with certainty there exists some probability distribution over P_1.[5]

The conditions which describe the solution to this problem and the analysis of the effect of changes in the negative income tax variables on individuals' decisions within the framework of this model are detailed in Appendix B.

The change in the equilibrium amount of labour devoted to producing Q_1, caused by an increase in the guarantee is given by

$$\frac{\partial L_1}{\partial G} = E\left[\frac{-|{}^{11}H|[U_{11}(1-t)P_1F_L^1 - U_{21}] + (1-t)U_{L_1L_3}U_1P_2F^2{}_{LL}[U_{11}(1-t)w - U_{21}]}{|H|}\right],$$

where H is the hessian formed from the elements $\langle U_{L_iL_j}\rangle$ i,j = 1,2,3, and the signs of all the terms except U_{21} and $U_{L_1L_3}$ are known and result from the model's assumptions. From this it is clear that strong behavioural assumptions, other than that the individual is averse to risk, are required when attempting to ascertain the sign of $\partial L_1/\partial G$. Although income and leisure are substitutes in the individual's budget, it seems reasonable to assume that they are complements in his utility function; and hence that $U_{21} > 0$. However, from Appendix B, Note (ii), $U_{21} > 0$ implies $U_{L_1L_3} < 0$, or that firm and wage work are substitutes in the individual's utility function, which suggests that the sign of $\partial L_1/\partial G$ is indeterminate.

When P_1 is uncertain or known with certainty an increase in the guarantee will not change the amount of work done in the production of Q_2. Indeed if P_1 was certain, changing the guarantee level would leave L_1 unaffected also.[6]

The impact of an increase in the guarantee on the amount of time spent working outside the firm is given by

$$\frac{\partial L_3}{\partial G} = (1-t)\ E\left[\frac{U_{L_3L_1}U_1P_2F^2_{LL}\ [U_{11}(1-t)P_1F^1_L - U_{21}] - U_{L_1L_1}U_1P_2F^2_{LL}[U_{11}(1-t)w - U_{21}]}{|H|}\right]$$

the sign of which is again dependent on $U_{L_1L_3}$ and U_{21} and is indeterminate. However, if P_1 is certain, the effect of an increase in the guarantee is given by

$$\frac{\partial L_3}{\partial G} = \frac{-U_1^2P_1F^1_{LL}P_2F^2_{LL}[U_{11}(1-t)w - U_{21}](1-t)}{|H|},$$

which implies that the amount of wage work will fall as the guarantee increases if income and leisure are complements in the utility function.

The effect of an increase in the tax rate is demonstrated in Appendix B. Again, within the context of the model the only determinate result is that the amount of labour used in the production of Q_2 will remain unchanged.

Work in the firm only. In this section it is assumed that the individual does not work outside the firm when he has chosen the amount of his labour which maximizes his expected utility of income and leisure and that the firm does not engage in the production of Q_2.

The change in the equilibrium amount of labour which the individual supplies to the firm, stemming from an increase in the guarantee level is described in Appendix C and is given by

$$\frac{\partial L_1}{\partial G} = \frac{-E[U_{11}(1-t)P_1F_L^1-U_{21}]}{D_2} ,$$

which is negative both under certainty and uncertainty if income and leisure are complements. The effect of changing the tax rate is indeterminate and results from the fact that an increase in the tax rate lowers income but also lowers the price of leisure.

An Interpretation

If, in the absence of uncertainty, the objective of a firm manager is simply to maximize his utility of income, and if the variable inputs are in perfectly elastic supply at their market prices, then the previous analysis suggests that the imposition of a negative income tax will not affect production decisions. However, it also implies that under output, or output price uncertainty, the levels of output and variable inputs will be influenced by the presence of a negative income tax. The magnitude and direction of this effect are determined by the firm manager's attitude to risk. The assumption that the entrepreneur's own labour is in perfectly elastic supply at a particular wage rate is untenable for those managers who work in their firms. The implications of this were explored by considering firm operators' income-leisure choices.

The models, which explicitly evaluate the income-leisure decisions of entrepreneurs, suggest that under certainty if an individual works in his firm and for wages, a change in the guarantee or the tax rate will affect only the hours he is willing to work in outside employment. This result follows naturally from the specification of the model. Labour will be supplied to the firm until its marginal value product falls to the wage rate, when outside employment will be sought. This situation is depicted in Figure 1. In this diagram the firm operator works in the firm until the marginal value product of his labour, MVP_L, equals his wage rate outside the firm and then he works for wages. In Figure 1 the firm owner works oa hours in the firm and ab hours for wages. Under certainty, the imposition of a negative income tax will not alter the equilibrium condition $MVP_L = w$, provided the firm owner continues to work in outside employment. In this instance, the guarantee and tax rate will simply affect the firm owner's labour supply curve; and hence the hours he works for wages. In the event that the individual reduces his outside employment to zero, he may then reduce his hours worked in the firm. This is borne out by a comparison of the previous discussion concerning an individual working in the firm only, and one who works both in and outside the firm.

The introduction of uncertainty significantly alters these conclusions. It has been established that a firm producing under uncertainty and exhibiting absolute risk aversion will expand its production as the guarantee increases. However, if the choice between income and leisure is explicitly evaluated, the expansion of output conflicts with the desire for more leisure as the guarantee increases.[7] For this reason, the net effect of an increase in the guarantee on the amount of work done in the firm may be unclear if uncertainty is admitted. However, Work Decisions of Self-employed Persons

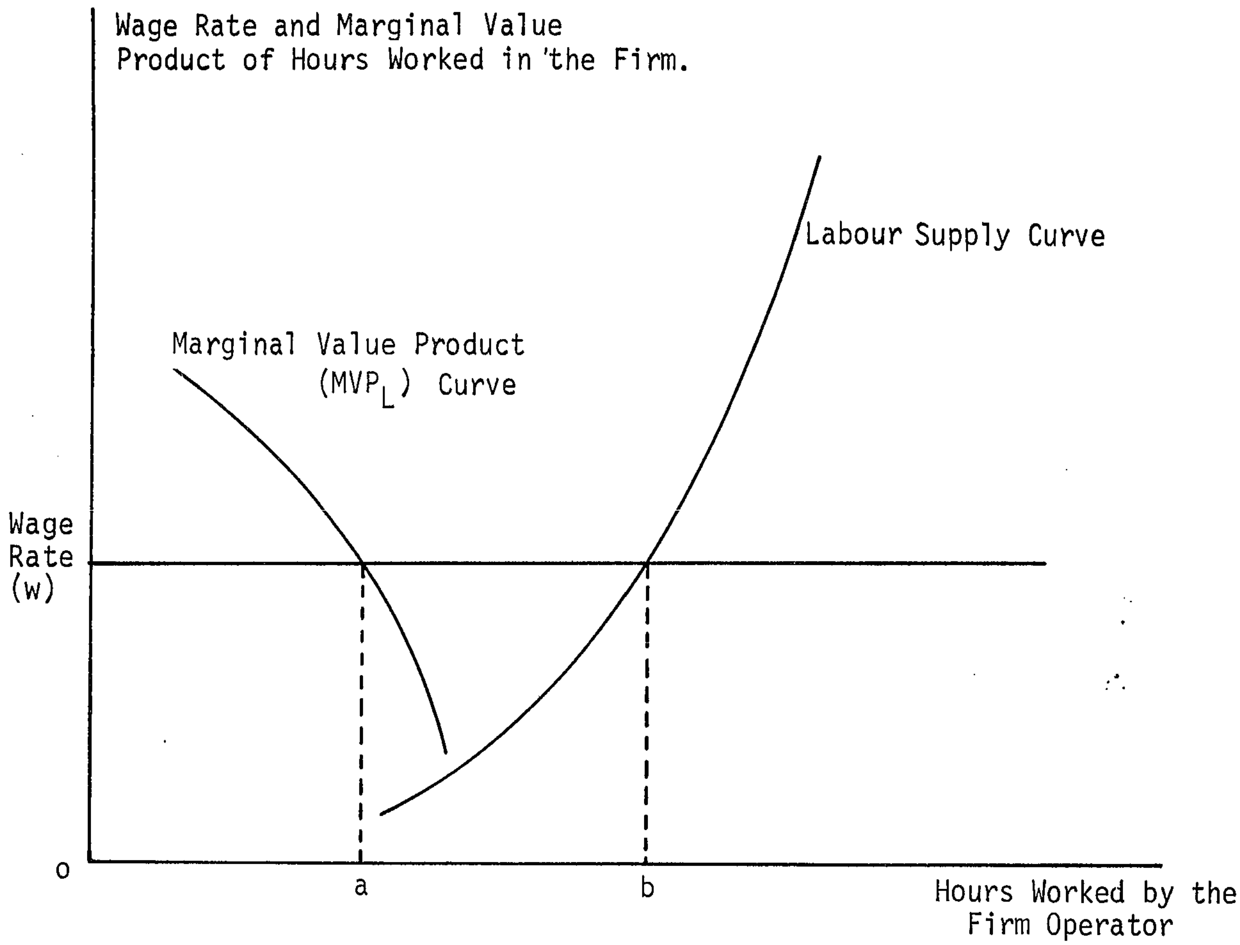

Figure 1.

if the firm operator works for wages before and after the imposition of a negative income tax plan, the amount of labour he supplies in the production of the output whose price is known with certainty, will not be affected by such a scheme.[8] For those firm operators who do not work for wages the foregoing model predicts that under certainty and uncertainty an increase in the guarantee will result in a reduction in the quantity of labour supplied to the firm by the firm operator. The effect of an increase in the tax rate on the firm operator's labour supply to the firm is not predictable except where he works for wages and uncertainty is ignored. In this instance a change in the tax rate associated with a negative income tax scheme will leave unchanged the quantity of labour the firm operator supplies to his firm.

The period of analysis is important for two reasons. First, it has been assumed throughout that the period of adjustment is short enough to enable capital to be regarded as a fixed input. It is possible that owner-operator firms would alter the quantity and quality of their stock of capital in response to the implementation of a negative income tax. This would affect the amount of labour supplied by the firm operator to the firm. Second, it has been assumed that the period of analysis is such that the operator's wage work competes directly with work in the firm. In this case the foregoing results suggest that the firm can be treated as if the firm owner's labour is in perfectly elastic supply at his wage rate. In the case of firms, with a marked seasonal demand for labour, an analysis period of a year may mean that observed hours of work outside the firm are not competing with work in the firm. If wage work does not compete with firm work then it may be appropriate to take cognisance of the effect of wage earnings on income and to implement the analysis of firm behaviour as if the operator worked in the firm only.

III. Conclusions

The imposition of a negative income tax scheme may, under certain conditions, reduce the riskiness of an income stream. This conclusion is a consequence of the result that a risk-averse individual would prefer a negative income tax plan even if his income under such a scheme had the same expected value as his income without the plan. The derivation of this result employs the tenuous assumption that the probability distribution of an individual's income would be unaffected by a negative income tax scheme.

The effect of a negative income tax plan on the decisions made by self-employed persons was also considered. It was postulated that those persons who manage, but do not work in, their firms obtain utility from income only. In the absence of uncertainty a negative income tax plan has no effect on managerial decisions. In contrast, in the presence of output price uncertainty the firm operator's attitude to risk determines the adjustments he makes to a negative income tax scheme. Indeed, a prediction of interest is that the firm operator may respond to a higher guarantee by choosing a higher level of output.

The analysis of operators who managed and worked in their firms was complicated by the possibility of wage work. These firm operators were posited to receive utility from both income and leisure. In the presence of uncertainty it was not possible to predict their response to a negative income tax scheme, unless they worked in their firms only. The elimination of uncertainty yielded models which predicted that only those farmers who did not work for wages before or after the imposition of a negative income tax plan would adjust their firm's production in response to such a scheme.

The models described above dealt with particular form of uncertainty; and hence the specific results obtained will not necessarily carry over into a more general framework. However, this analysis does suggest that hypotheses concerning the negative income tax, established from economic theory under certainty may be very different from those derived from a theory which admits uncertainty.

APPENDIX A

This analysis follows closely the procedure adopted by Sandmo [1971]. Assumptions:

(1) The individual's attitude to risk is summarised by a Von Neumann-Morgenstern utility function.

(2) The individual's objective is to maximize expected utility of income I.

(3) $U'(I) > 0$, $U''(I) < 0$, [risk aversion].

(4) There exists some probability distribution over the price of output.

(5) The firm's cost function is defined $F(X) = C(X) + B$; where $C(0) = 0$, $C'(X) > 0$ and B is fixed costs.

The firm's profit function is $\pi(X) = PX - C(X) - B$, and the N.I.T. payments are given by:

$$N = \begin{cases} G - t\pi(X) & \text{for } G \geq t\pi(X) \\ 0 & \text{for } G < t\pi(X) \\ G & \text{for } \pi(X) \leq 0 \end{cases} .$$

G - guarantee
t - tax rate

Then it follows that

$$\text{Income} = I(X) = N + \pi(X)$$
$$= G + (1-t)\pi(X) .$$

Assuming that $0 \leq t\pi(X) \leq G$, the individual's objective may be stated:

$$\max_X \; E[U(I(X))] = E[U(G + (1-t)(PX - C(X) - B))] .$$

The first order condition is

$$E[U'(I(X))(1-t)(P-C'(X))] = 0 , \qquad \text{(A1)}$$

and to ensure a maximum

$$D = E[U''(I(X))((1-t)(P-C'(X)))^2 - U'(I(X))(1-t)C''(X)] < 0 \; . \qquad \text{(A2)}$$

A. The Effect of an Increase in the Guarantee Level

Differentiating (A1) with respect to G yields

$$E[U''(I(X))(1-t)(P-C'(X))[1 + (1-t)(P-C'(X))\frac{\partial X}{\partial G}]$$

$$- U'(I(X))(1-t)C''(X)\frac{\partial X}{\partial G}] = 0 \; ,$$

which implies that

$$E[U''(I(X))(1-t)(P-C'(X)) + U''(I(X))(1-t)^2(P-C'(X))^2\frac{\partial X}{\partial G}$$

$$- U'(I(X))(1-t)C''(X)\frac{\partial X}{\partial G}] = 0 \; .$$

This expression can be written as

$$E[U''(I(X))(1-t)(P-C'(X))] + D\,\frac{\partial X}{\partial G} = 0 \; , \qquad \text{from (A2)}$$

and consequently the change in output caused by a change in the guarantee is given by

$$\frac{\partial X}{\partial G} = -\frac{1}{D}E[U''(I(X))(1-t)(P-C'(X))] \; . \qquad \text{(A3)}$$

Decreasing absolute risk aversion is a necessary and sufficient condition for $\frac{\partial X}{\partial G} \geq 0$ (see Note (1) part A in this Appendix).

B. The Effect of an Increase in the Tax Rate

Note that (A1) can be rewritten

$$E[U'(I(X))(P-C'(X))] = 0 \; , \qquad \text{(A6)}$$

by factoring out (1-t).

Differentiating A4 with respect to t yields

$$E[-U''(I(X))\pi(P-C'(X)) + U''(I(X))(1-t)(P-C'(X))^2\frac{\partial X}{\partial t} \cdot$$

$$-U'(I(X))C''(X)\frac{\partial X}{\partial t} = 0 \ ,$$

which implies

$$-E[U''(I(X))\pi(P-C''(X))] + \frac{D}{(1-t)} \ \frac{\partial X}{\partial t} = 0 \ .$$

Hence

$$\frac{\partial X}{\partial t} = \frac{(1-t)}{D} E[U''(I(X))\pi(P-C'(X))] \ , \qquad \text{(A5)}$$

which will be positive if relative risk aversion is increasing, otherwise it is indeterminate. (See Note (1) part B of this Appendix).

<u>Note (1)</u>

Absolute risk aversion is defined as $ARA \equiv \frac{-U''(I)}{U'(I)}$

Relative risk aversion is defined as $RRA \equiv \frac{-U''(I)I}{U'(I)}$

A. ARA, a decreasing function of I reflects the hypothesis that as the decision-maker becomes wealthier, his risk premium for any risky prospect should not increase.

Let $\bar{I}$ be the level of net revenue when $P = C'(X)$, then ARA decreasing implies

$$\frac{-U''(I)}{U'(I)} \leq ARA(\bar{I}) \text{ for } P - C'(X) \geq 0 \ . \qquad \text{(A6)}$$

Also, note that

$$-U'(I)(1-t)(P-C'(X)) \leq 0 \text{ for } P - C'(X) \geq 0, \text{ as } U' > 0 \ . \qquad \text{(A7)}$$

The multiplication of (A6) by the left hand side of (A7) yields

$$U''(I)(1-t)(P-C'(X)) \geq -ARA(\bar{I})U'(I)(P-C'(X))(1-t) \ .$$

This is true for all P for if $P \leq C'(X)$ the inequality in (A7) is reversed, but so is that in (A6).

Taking expected values

$$E[U''(I)(1-t)(P-C'(X))] \geq -ARA(\bar{I})E[U'(I)(1-t)(P-C'(X))]$$

$$\geq 0 \text{ from the first order conditions.}$$

This result and the sign of D confirms that

$$\frac{\partial X}{\partial G} \geq 0,$$

if the individual's utility function exhibits increasing absolute risk aversion.

B. Increasing relative risk aversion implies the hypothesis that if both wealth and the size of the bet are increased in the same proportion, the willingness to accept the bet will decrease.

RRA increasing implies

$$\frac{-U''(I)I}{U'(I)} \geq \frac{-U''(\bar{I})\bar{I}}{U'(\bar{I})} \qquad \text{for } P - C'(X) \geq 0 \ .$$

This inequality can be rewritten as follows:

$$\frac{-U''(I)}{U'(I)} \geq \frac{-U''(\bar{I})\bar{I}}{U'(\bar{I})I} \qquad \text{as } I \geq 0$$

$$\geq \frac{-U''(\bar{I})(G+(1-t)\bar{\pi})}{U'(\bar{I})(G+(1-t)\pi)} \qquad \text{for } P - C'(X) \geq 0, \qquad \text{(A8)}$$

using the definition of income and the assertion that profit is given by $\bar{\pi}$ when $P = C'(X)$.

Note that:

$$\frac{G + (1-t)\bar{\pi}}{G + (1-t)\pi} \geq \frac{\bar{\pi}}{\pi} \text{ for } P \geq C'(X)$$

$$\leq \frac{\bar{\pi}}{\pi} \text{ for } P \leq C'(X) \ , \tag{A9}$$

where the inequalities in (A9) result from the definition of $\bar{\pi}$ and the following

$$\bar{\pi} \leq \pi \text{ for } P \geq C'(X) \ ,$$

$$\bar{\pi} \geq \pi \text{ for } P \leq C'(X) \ .$$

Conditions (A9) and Equation (A8) imply that,

$$\frac{-U''(I)\pi}{U'(I)} \geq \frac{-U''(\bar{I})\bar{\pi}}{U'(\bar{I})} \text{ for } P - C'(X) \geq 0 \ . \tag{A10}$$

The multiplication of both sides of (A10) by $-U'(I)(P - C'(X))$ yields

$$U''(I)\pi(P - C'(X)) \leq \frac{U''(\bar{I})}{U'(\bar{I})} \bar{\pi} \, U'(I)(P - C'(X)) \ . \tag{A11}$$

The inequality (A11) holds for all P. The derivation of (A11) demonstrates that it holds for $P \geq C'(X)$. It remains valid for $P \leq C'(X)$ as in this case the inequality in (A10) is reversed but the sign of $-U'(I)(P - C'(X))$ is also reversed.

Taking the expectation over price through (A11) yields the following

$$E[U''(I)\pi(P-C'(X))] \leq \frac{U''(\bar{I})}{U'(\bar{I})} \bar{\pi} E[U'(I)(P-C'(X))]$$

$$\leq 0 \ .$$

This together with the sign of D means that if the firm manager's utility function exhibits increasing relative risk aversion output will increase as the tax rate increases. Application of the foregoing procedure suggests that the sign of $\frac{\partial X}{\partial t}$ is not predictable when the utility function is characterised by decreasing relative risk aversion.

Note (2)

The individual's objective is to:

$$\max_{L} E[U(I(L))] = E[U(G + (1-t)(PF(\bar{K},L) - wL - B))]$$

where

(a) There exists some probability distribution over $Q = F(K,L)$ given K and L,

(b) Capital is fixed at $\bar{K}$,

(c) $B = r\bar{K}$ where r is the cost of capital,

(d) L is the amount of labour supplied to the firm at wage rate w.

The first order condition is (eliminating (1-t))

$$E[U'(PF_L-w)] = 0 \ , \tag{A12}$$

and the second order condition is given by

$$D_1 = E[U''(PF_L-w)^2(1-t) + U'PF_{LL}] < 0 \ .$$

To ascertain the effect of changing the guarantee differentiate (A12) with respect to G. This yields

$$D_1 \frac{\partial L}{\partial G} + E[U''(PF_L-w)] = 0,$$

which can be rewritten

$$\frac{\partial L}{\partial G} = \frac{-E[U''(PF_L-w)]}{D_1} \ . \tag{A13}$$

Similarly the effect of a change in the tax rate is given by

$$\frac{\partial L}{\partial t} = \frac{E[U''(PF_L - w)\pi]}{D_1} \ . \tag{A14}$$

The proof that

(a) decreasing risk aversion implies $\frac{\partial L}{\partial G} \geq 0$,

(b) increasing relative risk aversion implies $\frac{\partial L}{\partial t} \geq 0$,

follows the procedure outlined in Note (1).

APPENDIX B

In this Appendix the model used to evaluate the reaction of self-employed persons to a negative income tax scheme embodies the following assumptions:

(i) $0 \leq y \leq G/t$,

(ii) the firm is composed of the two separate production functions

$$Q_1 = F^1(K_1,L_1), \qquad Q_2 = F^2(K_2,L_2),$$

$$F^i_{K_i} , F^i_{L_i} > 0, \ F^i_{L_iL_i} \ll 0, \ i = 1,2 ,$$

(iii) Earned income is given by wage earnings, wL_3, plus profits from the firm, π , or

$$y = P_1F^1(\bar{K}_1,L_1) + P_2F^2(\bar{K}_2,L_2) + wL_3 - r[\bar{K}_1 + \bar{K}_2]$$

$$= \pi + wL_3,$$

where

P_i, i = 1, 2 is the price of the ith output,

w is the wage rate, r is the cost of capital,

$L_1 + L_2$ the hours the owner supplies to the firm,

L_3 the hours the firm owner works for employers outside the firm,

π is profit,

(iv) There exists a probability distribution over P_1,

(v) The individual has to make the short run choice concerning L_1, L_2 and L_3; the capital inputs K_1 and K_2 are assumed to be fixed at $\bar{K}_1$ and $\bar{K}_2$ respectively.

(vi) The individual is assumed to be averse to risk. That is the utility function is assumed to be concave.

Including the trade-off between leisure and earned income, the individual's objective is to:

$$\underset{L_1\, L_2\, L_3}{\text{Maximize}}\ E[U(G + (1-t)(P_1F^1(\bar{K}_1,L_1) + P_2F^2(\bar{K}_2,L_2) + wL_3 - B),$$

$$\bar{L} - L_1 - L_2 - L_3)],$$

where

$B = r(\bar{K}_1 + \bar{K}_2)$,

$\bar{L}$ is the maximum hours the individual can work.

Using common terminology, the first order conditions are:

$$E[U_{L_1}] = E[U_1(1-t)P_1F^1_L - U_2] = 0 \qquad \text{(B1)}$$

$$E[U_{L_2}] = E[U_1(1-t)P_2F^2_L - U_2] = 0 \qquad \text{(B2)}$$

$$E[U_{L_3}] = E[U_1(1-t)w - U_2] = 0 \qquad \text{(B3)}$$

where

$F^i_L = F^i_{L_i} = \partial F^i/\partial L_i \quad i = 1, 2,$

$U_1 = \partial U/\partial I,$

$U_2 = \partial U/\partial(\bar{L} - L_1 - L_2 - L_3),$

$U_{L_i} = \partial U/\partial L_i \quad i = 1, 2, 3.$

Note (i) (B2) and (B3) imply that $P_2F^2_L = w$ as P_2, $F^2_{L_2}$ and w are not stochastic.

Note (ii) $U_{L_1L_1} = U_{11}((1-t)P_1F_L^1)^2 + U_1(1-t)P_1F_{LL}^1 - 2U_{21}P_1F_L^1(1-t) + U_{22}$

$$U_{L_1L_3} = U_{11}(1-t)^2P_1F_L^1w - U_{21}w(1-t) - U_{12}P_1F_L^1(1-t) + U_{22}$$

$$U_{L_1L_2} = U_{11}(1-t)^2P_1F_L^1P_2F_L^2 - U_{21}P_2F_L^2(1-t) - U_{12}P_1F_L^1(1-t) + U_{22}$$

$$U_{L_2L_2} = U_{11}((1-t)P_2F_L^2)^2 + U_1(1-t)P_2F_{LL}^2 - 2U_{21}P_2F_L^2(1-t) + U_{22}$$

$$U_{L_2L_3} = U_{11}(1-t)^2P_2F_L^2w - U_{12}(1-t)P_2F_L^2 - U_{21}(1-t)w + U_{22}$$

$$U_{L_3L_3} = U_{11}((1-t)w)^2 - 2U_{21}(1-t)w + U_{22}$$

Note (iii) The fact that $U_{L_iL_j} = U_{L_jL_i}$ j, i = 1, 2, 3

and the result $P_2F_L^2 = w$ imply

$$U_{L_3L_2} = U_{L_3L_3} = U_{L_2L_3} \text{ and } U_{L_1L_3} = U_{L_1L_2} = U_{L_3L_1} = U_{L_2L_1}\,.$$

Note (iv) It is assumed that (B1), (B2) and (B3) describe an interior solution. To ensure (B1), (B2) and (B3) describe a maximum it is required that:

(a) $E[U_{L_iL_i}] < 0 \quad i = 1, 2, 3,$

(b) $E[|{}^{ii}H|] > 0$, i = 1, 2, 3; where H^{ij} is the matrix formed by deleting the ith row and jth column of the matrix H,

(c) $E[|H|] < 0$; where H is the hessian formed from the $\{U_{L_iL_j}\}$ i, j = 1, 2, 3.

The stronger assumption that $U_{L_1L_1}, U_{L_2L_2}, U_{L_3L_3} < 0, |{}^{ii}H| > 0$, i = 1, 2, 3, $|H| < 0$ is also made. This in turn implies that (a), (b) and (c) hold.

To ascertain the effect on L_1, L_2 and L_3 of varying the guarantee or the tax rate differentiate equations (B1), (B2), and (B3) totally.

$$E\left[\begin{bmatrix} U_{L_1L_1} & U_{L_1L_2} & U_{L_1L_3} \\ U_{L_2L_1} & U_{L_2L_2} & U_{L_2L_3} \\ U_{L_3L_1} & U_{L_3L_2} & U_{L_3L_3} \end{bmatrix} \begin{bmatrix} dL_1 \\ dL_2 \\ dL_3 \end{bmatrix} = \begin{bmatrix} -[U_{11}(1-t)P_1F_L^1-U_{21}] & [U_{11}(1-t)P_1F_L^1-U_{21}]\pi + U_1P_1F_L^1 \\ -[U_{11}(1-t)P_2F_L^2-U_{21}] & [U_{11}(1-t)P_2F_L^2-U_{21}]\pi + U_1P_2F_L^2 \\ -[U_{11}(1-t)w-U_{21}] & [U_{11}(1-t)w-U_{21}]\pi + U_1w \end{bmatrix} \begin{bmatrix} dG \\ dt \end{bmatrix}\right]$$

Using Cramer's rule, the effect on L_1 of an increase in the guarantee is given by

$$\frac{\partial L_1}{\partial G} = E\left[\frac{-|{}^{11}H|[U_{11}(1-t)P_1F_L^1-U_{21}]+|{}^{21}H|[U_{11}(1-t)P_2F_L^2-U_{21}]-|{}^{31}H|[U_{11}(1-t)w-U_{21}]}{|H|}\right]$$

$$= E\left[\frac{-|{}^{11}H|[U_{11}(1-t)P_1F_L^1-U_{21}]+[|{}^{21}H|-|{}^{31}H|][U_{11}(1-t)w-U_{21}]}{|H|}\right], \qquad \text{(B4)}$$

as $P_2F_L^2 = w$.

By direct computation

$$|{}^{21}H|-|{}^{31}H| = U_{L_1L_2}U_{L_3L_3} - U_{L_3L_2}U_{L_1L_3} - U_{L_1L_2}U_{L_2L_3} + U_{L_2L_2}U_{L_1L_3}$$

$$= U_{L_1L_3}\; U_1P_2F_{LL}^2(1-t) ,$$

using Notes (ii) and (iii). Consequently (B4) can be written

$$\frac{\partial L_1}{\partial G} = E\left[\frac{-|{}^{11}H|[U_{11}(1-t)P_1F_L^1-U_{21}]+(1-t)U_{L_1L_3}U_1P_2F_{LL}^2[U_{11}(1-t)w-U_{21}]}{|H|}\right] . \qquad \text{(B5)}$$

If the marginal utility of income (leisure) is unaffected by, or increased by an increase in leisure (income), that is $U_{21} \geq 0$, then the sign of the term within the brackets of Equation (B5), and hence the sign of $\frac{\partial L_1}{\partial G}$ will be determinate only if $U_{L_1L_3} \geq 0$.

Note (v). Under certainty $P_1F^1_L = P_2F^2_L = w$ and Equation (B5) reduces to

$$\frac{\partial L_1}{\partial G} = \frac{[|^{21}H|-|^{31}H|-|^{11}H|][U_{11}(1-t)w-U_{21}]}{|H|} \tag{B6}$$

$$= 0,$$

as $|^{21}H|-|^{31}H|-|^{11}H| = 0.$

The effect on L_2 of increasing the guarantee level is given by

$$\frac{\partial L_2}{\partial G} = E\left[\frac{|^{12}H|[U_{11}(1-t)P_1F^1_L-U_{21}]+[|^{32}H|-|^{22}H|][U_{11}(1-t)w-U_{21}]}{|H|}\right], \tag{B7}$$

using the fact that $P_2F^2_L = w$.

After rearranging $|^{32}H|-|^{22}H|$ can be expressed

$$|^{32}H|-|^{22}H| = U_{L_1L_1}[U_{L_2L_3}-U_{L_3L_3}] + U_{L_1L_3}[U_{L_1L_3}-U_{L_2L_1}]$$

$$= 0 ,$$

using Note (iii). Also from Note (iii) the determinant of ^{12}H is

$$|^{12}H| = U_{L_2L_1}\,U_{L_3L_3} - U_{L_3L_1}\,U_{L_2L_3}$$

$$= 0.$$

Substituting these results in Equation (B7) yields the conclusion that $\frac{\partial L_2}{\partial G} = 0$.

The effect of an increase in the guarantee level on work done outside the firm is given by

$$\frac{\partial L_3}{\partial G} = E\left[\frac{-|{}^{13}H|\,[U_{11}(1-t)P_1F_L-U_{21}]+[|{}^{23}H|-|{}^{33}H|\,][U_{11}(1-t)w-U_{21}]}{|H|}\right], \quad \text{(B8)}$$

again using the fact that $P_2F_L^2 = w$. The following results are obtained by direct computation and the use of Notes (ii) and (iii):

$$[|{}^{23}H|-|{}^{33}H|\,] = U_{L_1L_1}\,U_{L_3L_2} - U_{L_3L_1}\,U_{L_1L_2} - U_{L_1L_1}\,U_{L_2L_2} + U_{L_1L_2}\,U_{L_1L_2}$$

$$= -U_{L_1L_1}\,U_1P_2F_{LL}^2(1-t)\ , \quad \text{(B9.1)}$$

$$|{}^{13}H| = U_{L_2L_1}\,U_{L_3L_2} - U_{L_2L_2}\,U_{L_3L_1}$$

$$= -U_{L_3L_2}\,U_1P_2F_{LL}^2(1-t)\ . \quad \text{(B9.2)}$$

The substitution of (B9.1) and (B9.2) in (B8) yields

$$\frac{\partial L_3}{\partial G} = E\left[\frac{(1-t)U_{L_3L_1}U_1P_2F_{LL}^2[U_{11}(1-t)P_1F_L^1-U_{21}]-(1-t)U_{L_1L_1}U_1P_2F_{LL}^2[U_{11}(1-t)w-U_{21}]}{|H|}\right].$$

Again, if it is assumed that $U_{21} \geq 0$, the sign of $\frac{\partial L_3}{\partial G}$ is negative if $U_{L_3L_1} \geq 0$, otherwise it is indeterminate.

Following the procedure outlined above the effects on L_1, L_2 and L_3 respectively of an increase in the tax rate are:

$$\frac{\partial L_1}{\partial t} = E\left[\frac{|^{11}H|([U_{11}(1-t)P_1F_L^1-U_{21}]\pi+U_1P_1F_L^1)-(1-t)U_{L_1L_3}U_1P_2F_{LL}^2([U_{11}(1-t)w-U_{21}]\pi+U_1w)}{|H|}\right], \tag{B10}$$

$$\frac{\partial L_2}{\partial t} = 0 , \tag{B11}$$

$$\frac{\partial L_3}{\partial t} = E\left[\frac{U_{L_3L_1}\ U_1P_2F_{LL}^2([U_{11}(1-t)w-U_{21}]\pi+U_1w) - U_{L_1L_1}U_1P_2F_{LL}^2([U_{11}(1-t)P_1F_L^1-U_{21}]\pi+U_1P_1F_L^1)}{|H|}\right](1-t) \tag{B12}$$

The signs of $\partial L_1/\partial t$ and $\partial L_3/\partial t$ are not predictable from the expressions presented in (B10) and (B12) respeetively.

APPENDIX C

Embodying the assumptions established in Appendix B but excluding wage work and the production of Q_2 from the model, the individual's objective is assumed to be

$$\underset{L_1}{\text{Maximize}}\ E[U(G + (1-t)(P_1F^1(\bar{K}_1,L_1) - B),\ \bar{L} - L_1)$$

The first order condition is

$$E[U_1(1-t)P_1F_L^1 - U_2] = 0\ , \qquad (C1)$$

and the second order condition is given by

$$D_2 = E[U_{11}((1-t)P_1F_L)^2 - U_{12}(1-t)P_1F_L^1 + U_1(1-t)P_1F_{LL}^1 - U_{21}(1-t)P_1F_L^1 + U_{22}] < 0\ .$$

The effect on L_1 of an increase in the guarantee is obtained by differentiating (C1) with respect to G. This yields

$$D_2\frac{\partial L}{\partial G}1 + E[U_{11}(1-t)P_1F_L^1 - U_{21}] = 0,$$

and which can be written

$$\frac{\partial L}{\partial G}1 = \frac{-E[U_{11}(1-t)P_1F_L^1 - U_{21}]}{D_2}\ . \qquad (C2)$$

Similarly the effect on L_1 of an increase in the tax rate is given by

$$\frac{\partial L}{\partial t}1 = \frac{E[U_1P_2F_L^2 + (U_{11}P_1F_L^1(1-t) - U_{21})\pi]}{D_2} \qquad (C3)$$

Although the assumption that $U_{12} \geq 0$ means that $\partial L_1/\partial G < 0$, it does not resolve the sign of the expression in (C3).

FOOTNOTES

[1]For a detailed discussion of the Von Neumann-Morgenstern utility function and the axioms which ensure its existence, see De Groot (1970).

[2]Throughout this paper the partial derivative of any function, say $g(x_1,x_2)$ is denoted $\frac{\partial g}{\partial x_1} = g_{x_1}$, or $\frac{\partial g}{\partial x_1} = g_1$. The partial derivative of a function which has one argument only, say $h(x)$, is denoted $\frac{\partial h(x)}{\partial x} = h'(x)$.

[3]A detailed example of this is provided by McCall (1971).

[4]The validity of this conclusion is apparent from Equations (A13) and (A14) in Appendix A, Note (2), and the fact that $PF_L = W$ in the absence of uncertainty.

[5]The results which follow would also be obtained if P_1 was assumed to be known with certainty but $Q_1 = F(K_1,L_1)$ was uncertain given L_1 and K_1.

[6]This result is demonstrated in Appendix B Note (v).

[7]This statement assumes that leisure is a normal good.

[8]This result is a consequence of the assumption that the firm's two outputs, Q_1 and Q_2, are produced by separate production functions.

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