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STOCHASTIC MODELS FOR PENSIONABLE SERVICE

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ABSTRACT

This paper investigates the effects of vesting rules and turnover rates on the accumulation of pensionable service. First, a theoretical framework is developed, through stochastic models in continuous time, to characterize the cumulative pensionable service with reference to its moments, under an arbitrary age-service vesting rule. In the most general case, the rate of employment termination is allowed to be a function of both age and time spent in an employment. Next, a discrete computational model is presented for the determination of the distribution of pensionable service and of related measures, using age and service-dependent termination rates as input. This model is applied to Canadian Public Service and the results are discussed. Also considered are the employer's perspective and passage from pensionable service to pension benefits.

Stochastic Models for Pensionable Service

1. INTRODUCTION

As a means of promoting general welfare, the practice of providing pensions for the retired has developed into a rapidly growing industry in North America in the 20th century, especially since World War II. This growth is landmarked by large-scale federal undertakings like the Social Security Act in the United States and the Canada Pension Plan in Canada, by the entrance into the industry of trust and insurance companies to offer contracts for private industry pension plans, and by the legal identification of pensions as proper subjects for collective bargaining. Private pension plans now cover about 33 million persons in the United States alone, representing 44 percent of the entire work force in commerce and industry [1].

In general, pension plans may be divided into three categories. In the first category, the ensuing benefits are related to the salary and length of service of the employee. In the second category, the pension is fixed or it relates only to amounts of contributions or to the years of service. In the third category, the contributions alone determine the benefits. The first type is the most common one and is used for the most part for salaried employees. Pensions for hourly workers are usually calculated in terms of dollars per month per year of service. Even when the pension is related primarily to earnings, the common practice is to assign a percentage to each year of service so that the length of service is also taken into account.

The main difficulty for all concerned in assessing the value and/or cost of a pension plan lies in the peculiar method used historically to establish the ensuing benefits. In general, an individual who leaves an organization is entitled to pension benefits starting at retirement age if he meets two requirements: i) he is of a prescribed minimum age at the time of his leave (the age requirement) and ii) he has completed a prescribed minimum number of years of service with the organization (the service requirement). If both of these conditions are met, the pension is said to be vested and the individual collects benefits from it upon retirement, even if he never again works for that organization. The pension so vested is proportional to the number of years of service, generally referred to as the pensionable service or qualifying service. As an example, suppose that a person starts his working life at age 20, changes employment at age 40, then at 48, again at 60, and retires at 65. With the age-service requirements of 45-10, he would be entitled to only 12 pensionable years. If the requirements were 45-8 or 40-10, he would have been entitled to 20 or 32 pensionable years, respectively.

The service requirement is more predominant in private pension plans in North America, a majority of which involve 5 to 15 years of service to complete vesting. An examination of the vesting provisions in the United States in 1969 indicates that in over 99 percent of the plans a worker had to make at least a 5-year commitment to a firm in order to qualify for a pension. In addition, almost half the covered workers had to fulfill an age requirement [3]. In what follows, we shall symbolize the qualification requirement(s) either by [s], which will be called the simple vesting rule, or, if an age requirement must also be met, by the pair [a,s], which will be referred to as the composite vesting rule. For clarity, a will be measured from the beginning of the working life; thus in the preceeding example the three vesting rules considered would be [25,10], [25,8] and [20,10]. Note that in a composite vesting rule $a > s$ must hold for the age requirement to be operational.

The example above illustrates the sensitivity of the pension benefits to the vesting rules. Another important factor is the turnover rate which is related to completed lengths of service in different jobs. This factor underlies the stochastic nature of the problem and of the models to be proposed in the sequel. There are other elements that are also significant to any pension plan. The number of eligible persons, estimates of their earnings, secondary pension benefits to widow(er) or children, the current level of investment income that can be expected from the accumulated money, indexing, optional vesting, transferability, etc., all affect the accumulation of an adequate fund. This in turn has been the focal point of actuarial analyses regarding pension security, such analyses being directed to the estimation of annual contributions required to develop a satisfactory reserve fund to meet future liabilities. In this context, the vesting rule has been regarded mainly as an instrument for the firms - the age requirement defers membership in the plan for new employees, decreasing the short run administrative costs, and the service requirement denies the worker nonforfeitable benefits for an additional period, reducing the employee turnover in the longer run.

From the employee's point of view, both of these requirements have the effect of prolonging his obligation to the firm and therefore increasing the risk to him. This and similar considerations have made the private pension industry the center of much public debate in North America in recent years. In 1974, a pension reform law, the Employee Retirement Income Security Act, was enacted in the United States which limits the service requirement to 15 years. At present, six provinces in Canada have enacted legislation regarding vesting provisions. The effects of vesting rules and turnover rates on the accumulation of pensionable service, however, remain unclear. In what follows, we shall be concerned with this problem. We shall develop continuous and discrete time stochastic models to predict the average statistical behaviour of a group of similar workers, in terms of the cumulative pensionable service over time, as a function of the vesting rules and completed lengths of service or termination rates. Such predictions would be instrumental in planning for the future: a) for individuals to determine their optimal levels of saving for retirement, b) for employers to better assess the costs of pension benefits, and c) for the governments to forecast costs of their retirement programs, as these costs are related to the income of the aged.

The problem can be described as follows. Let $X(t)$ be the length of service in the present employment, where t represents the time since the beginning of the working life. For simplicity, we assume that a person who terminates an employment starts in a new one right away. Let there be n employment terminations in the interval $(0,T]$ and let t_i $(i = 1,2,\ldots,n)$ be the instant just before the i-th termination $(t_n \equiv T)$. **Let $X(t_i)$ represent a completed length** of service (CLS) that will be added to pensionable service, under the composite vesting rule $[a,s]$, provided that $t_i \geq a$ and $X(t_i) \geq s$. That is, on letting

$$Y(t_i) = \begin{cases} X(t_i) & \text{if } t_i \geq a \text{ and } X(t_i) \geq s \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

the cumulative pensionable service over $(0,T]$, $\alpha_{as}(t)$, can be written as

$$\alpha_{as}(T) = \sum_{i=1}^{n} Y(t_i). \tag{2}$$

In what follows, we shall be concerned with the characterization of the random variable $\alpha_{as}(T)$. We denote the related n-th moment by $\mu_{as}^{(n)}(T)$ and drop the first subscript in symbols if the vesting rule is simple.

In section 2, we first develop a continuous time model with time-varying CLS distributions which allow for the dependency of termination rates both on time (age) and length of service. Special cases and extensions of this model are then presented from the points of view of both employees and employers. Section 3 is devoted to a discrete time computational model and its application to Canadian Public Service. Finally, in section 4, we present some conclusions and comment on applications.

2. CONTINUOUS TIME MODELS

In this section, we assume that completed lengths of service during an individual's working life form a sequence of independent random variables, with time varying distribution functions. More specifically, let Z_t be the length of a completed employment that commences at time t. Then

$$P[Z_t \leq x] = G_t(x), \ t \geq 0, \ x \geq 0. \qquad (3)$$

We take $G_t(x)$ to be continuous in x and t with density $g_t(x)$.

Note that the possible dependencies of termination rates on age and on the time spent in the present job are both accounted for in (3) (see section 2.3). Continuity assumptions on $G_t(x)$ are not restrictive; they are made primarily for convenience and in view of the fact that a job change may occur at any time. An independence assumption on the sequence of completed lengths of service is restrictive. Although there is no empirical evidence one way or the other, the sequence of CLS distributions would be dependent on total qualifying service to date, as well as on age. This would lead to another sequential corrolation between successive lengths of service, more pronounced, conceptually, near the age of retirement. These influences are assumed away with the contention that the allowed dependencies on age and length of service provide sufficient flexibility in describing the employment termination process with admissible accuracy in most circumstances. It should also be mentioned that actuarial data in the subject matter area is limited for the most part to rates of termination at different ages, and the requirements of (3) in terms of data might already be excessive.

Within the above framework, points of employment termination in an individual's working life form a nonhomogeneous renewal process over $(0,T]$. We now define the following densities:

$h(t)\Delta t = P[$an employment termination during $(t,t+\Delta t)]$

$h^*(t)\Delta t = P[$a pensionable employment termination during $(t,t+\Delta t)$ under the simple vesting rule$]$

We have

$$h(t) = g_0(t)+\int_0^t h(x)g_x(t-x)dx \tag{4}$$

and

$$h^*(t) = g_0(t) + \int_0^{t-s} h(x)g_x(t-x)dx, \quad t \geq s. \tag{5}$$

The first equation, which follows easily from the first principles, was studied first by Bartholomew ([2], pp. 224-230) in relation to a recruitment model. For in order to have an employment termination at about t, either the first employment must have lasted until about t or there must have been a termination at about x followed by a completed service of length about t-x. Bartholomew proposes the following approximate solution for (4):

$$h(t) \simeq g_0(t)+\frac{G_0(t) \int_0^t g_x(t-x)dx}{\int_0^t [1-G_x(t-x)]dx}, \tag{6}$$

which he claims is reasonably accurate under certain conditions. Equation (5) is derived through a similar argument. In order to have a pensionable employment termination at about t under the simple vesting rule, either the initial employment must have terminated at about t ($t \geq s$) or there must have been a termination at about x ($x \leq t-s$) followed by a completed service of about length t-x.

Based on $h^*(t)$, the expected cumulative pensionable service under the vesting rule [s] or [a,s] can be derived by deducing and solving a differential equation (see Appendix 1 for details). It turns out that

$$\mu_s(T) = T[1-G_0(T)] + \int_s^T xh(T-x)[1-G_{T-x}(x)]\,dx + \\ + \int_{u=s}^{T} u[g_0(u) + \int_{x=0}^{T-u} h(x)g_x(u)dx]\,du \qquad (7)$$

and

$$\mu_{as}(T) = \mu_s(T) - \int_{u=s}^{a} u\,[g_0(u) + \int_{x=0}^{a-u} h(x)g_x(u)dx]du \qquad (8)$$

where h(t) is given formally as the solution of (4) and approximately by (6).

An important special case of the above model arises when the functional form of the CLS distribution remains the same but its parameters vary over time. Results for some other special cases and extensions of interest are outlined below.

2.1. Different Initial CLS Distribution

Suppose that the distribution of the first completed length of service is different, say $G_1(x)$, but that those of all the others are the same, $G(x)$. This variation is of some practical interest due to the well-known tendency for the probability of turnover to decline with age and length of tenure, the first employment constituting, in general, the most imperfect "match" in terms of the worker-firm relationship. Now the renewal density $h(t)$ can be determined explicitly for a wide class of densities $g(x)$, $g_1(x)$, and we have:

$$\mu_s(T) = s[1-G_1(s)] + s[1-G(s)]\, H_1(T-s) + \int_s^T [1-G_1(u)]du + \int_s^T H_1(T-u)[1-G(u)]du, \text{ and} \tag{9}$$

$$\mu_{as}(T) = \mu_s(T) - \int_s^a u[g_1(u) + g(u)\, H_1(a-u)]du \tag{10}$$

where $H_1(T)$ is the renewal function related to the underlying "delayed" renewal process of employment terminations.

2.2. Stationary CLS Distributions

We now consider a further specialization of the general model and assume that the completed lengths of service during an individual's working life form a sequence of independent, identically distributed random variables with distribution function $G(x)$. Time invariance of CLS distributions is generally assumed in renewal theoretic models for graded social systems and for recruitment and wastage in an organization (see [2], Chapters 7 and 8). In these contexts, where each leaving employee is replaced by a person of similar age, tenure, and qualifications, the above assumption may be regarded as an adequate representation of reality. In the present context, with regard to the statistical behavior over time of a group of similar employees, time invariance of G(x) is objectionable. Other things being equal, we expect a decrease in an individual's propensity to resign from his later jobs. Still, the assumption provides a testable hypothesis and the model presented below will be useful in some extensions to follow.

With the assumption of this section, points of employment termination form an ordinary renewal process over (0,T]. Under the simple vesting rule [s], if u is the first employment termination point, we can write for $T \geq s$ that

$$\alpha_s(T) = \begin{cases} \alpha_s(T-u) & \text{if } u < s \\ u + \alpha_s(T-u) & \text{if } s \leq u \leq T \end{cases} \tag{11}$$

This relationship can be developed to determine all the moments related to the cumulative pensionable service recursively (see Appendix 1). The first moments, for example, are given by

$$\mu_s(T) = s[1-G(s)][1+H(T-s)] + \int_s^T [1+H(T-u)][1-G(u)]du \tag{12}$$

and

$$\mu_{as}(T) = \mu_s(T) - \int_s^a [1+H(a-u)]\, ug(u)\, du \tag{13}$$

where H(t) is the associated renewal function.

2.3. Strictly Time-Dependent Rate of Termination

If, as in the general model, the CLS distributions are allowed to be time-varying, the rate of termination (propensity to leave) at time t+x for an employment that commences at time t is given by

$$\lambda(x,t) = \frac{g_t(x)}{1-G_t(x)}, \; t \geq 0, \; x \geq 0 \tag{14}$$

Thus the rate of termination is a function of both the total length of employment (or age) and the time spent in the present employment. If $G_t(x) \equiv G(x)$, and $t \geq 0$, then $\lambda(x,t) = \lambda(x)$ and we have the model of section 2.2. Since in many cases the actuarial data pertains to rates of termination at different ages, the other extreme case of $\lambda(x,t) \equiv \lambda(x+t)$ has some practical significance. Suppose, therefore, that $\lambda(t)$, $t \geq 0$, is given. Within the framework of the general model, we first construct the CLS distribution by solving (14) with $\lambda(x,t) \equiv \lambda(x+t)$ to obtain

$$G_t(x) = 1 - \exp\left(-\int_0^x \lambda(u+t)\, du\right), \quad t \geq 0, \; x \geq 0 \tag{15}$$

and

$$g_t(x) = \lambda(t+x) \exp\left(-\int_0^x \lambda(u+t) du\right), \; t \geq 0, \; x \geq 0. \tag{16}$$

For convenience, we take $t = 0$ as the start of the working life, as before. It is easy to see that $h(t) = \lambda(t)$, $t \geq 0$, and, on using these in (7) we obtain, after some simplifications:

$$\mu_s(T) = sR(s) + \int_s^T R(u)du + \int_{x=0}^{T-s} \lambda(x)R(x)^{-1}[sR(s+x) + \int_{u=s}^{T-x} R(u+x)du]dx \tag{17}$$

where

$$R(t) = \exp\left(-\int_0^t \lambda(u)du\right). \tag{18}$$

The expression for $\mu_{as}(T)$ can also be easily derived.

2.4. Prediction at Time a

Consider an individual who has just reached the age barrier a, after having spent the time z in his present employment. Time to next employment termination, being a residual length of service, would have a different distribution, say $G_1(x)$, from the subsequent (if any) completed lengths of service. The predictive period is now (a,T] (or (0,T-a]) with the previous employment measures being deterministic. $G_1(x)$ is not being regarded as the forward recurrence time distribution at time a in the renewal process of employment terminations; rather, it is taken as the distribution of the first length of service, which has different statistical properties. This differentiation may also serve to accomodate a sudden change in termination rates after the age requirement is met. Such a change is anticipated implicitly by the vesting rules of a pension plan, as these rules are supposed to reduce employee turnover for new entrants. There is also empirical evidence to the effect that the vesting provisions may place a strong restraint on quit decisions before vesting, particularly for young entrants [6].

If we denote by $\overline{\mu}_{as}(T)$ the expected cumulative pensionable service over (a,T] (or(0,T-a]) under the above framework, we can write

$$\overline{\mu}_{as}(T) = \begin{cases} z + \int_0^{T-a} [u+\mu_s(T-a-u)]g_1(u)du & \text{if } z \geq s \\ \int_0^{s-z} \mu_s(T-a-u)g_1(u)du + \int_{s-z}^{T-a} [z+u+\mu_s(T-a-u)]g_1(u)du, & \text{if } z < s \end{cases} \tag{19}$$

where $\mu_s(t)$ is given by (12) in the homogeneous case or by (7) in the nonhomogeneous case.

2.5. Employer's Perspective

The above models are built to predict the cumulative pensionable service as a measure of the ultimate benefit to be derived by a group of similar workers from their career membership in pension plans. In this context, the accumulation of vested pension is affected through a sequence of qualifying lengths of service in different employments. On the other hand, firms in the private pension market, in addition to the benefits side, are primarily concerned with the accumulation of adequate funds to meet future liabilities. Central to an assessment of these liabilities and their implications as to the specific provisions of a pension plan (e.g., funding, benefits structure, qualification requirements, etc.) would again be the estimation of cumulative pensionable service over time. However, the process of accumulation in this context would be through the qualifying services of successive employees occupying a given position in the firm.

Consider, for example, a single position in an organization of constant size where each loss must be compensated by a gain. On separation, an incumbent is replaced by a new employee of comparable qualifications. The sequence of employment terminations over time again forms a point process, and under different assumptions on this point process we arrive at basically the same models and results as before, provided that the vesting rule is simple. The end of the predictive period, however, does not now correspond to the time of retirement; rather, $\mu_s(t)$, $t \geq 0$, for example, is the expected cumulative pensionable service at time t related to the position under consideration. Since in this case the job and the firm remain the same but the employees are replaced, the stationarity assumption of section 2.2. on the CLS distributions is more adequate and G(x) is easier to construct. Also, the limiting result

$$\mu_s = \lim_{t \to \infty} \frac{\mu_s(t)}{t} = \frac{1}{\gamma} [s[1-G(s)] + \int_s^\infty [1-G(u)]\, du], \quad s \geq 0, \tag{20}$$

which is obtained from (12), is a useful measure of the per unit time contribution of this position to pensionable service over an extended period of time, the position being characterized by the CLS distribution G(x) with mean γ.

If the vesting rule is composite, the age profile of the new entrants must also be taken into consideration. This requires a somewhat different approach and will not be discussed here.

The models presented in this section are formulated in continuous time to provide a theoretical framework for the prediction of pensionable service with reference to its moments. The emphasis has been on the analytic side, with distribution function(s) of completed lengths of service being the main input requirement. In a given application, these distribution functions would have to be specified by fitting one of the several functional forms available to empirical data on termination rates or lengths of service. Bartholomew, for example, reported considerable success in graduating CLS data through the mixed exponential distribution

$$G(x) = \sum_{i=1}^{n} p_i(1-e^{-\lambda_i x}), \quad \sum_{i=1}^{n} p_i = 1, \tag{21}$$

which provides considerable flexibility and simplifies computations (see [2], pp. 181-198 for a discussion of CLS distributions in general and of (21) in particular for $n = 2$ and $p_1, p_2 > 0$.)

The main intended use of the above models, however, is not so much computational as it is conceptual. Almost invariably, actuarial data on completed lengths of service is available in tabular form in terms of rates of termination by age, and in some cases by age and service. Discrete time models which are capable of using the input data directly in this format would facilitate applications in actual practice while permitting easier computations. The next section is devoted to the development and application of such a model.

3. A DISCRETE MODEL AND ITS APPLICATION

In this section, we denote by X_n the number of years of service for an individual in his current job at the beginning of his n-th year of employment. The first year of employment corresponds to the beginning of the working life, which in turn is taken to be of length N years. We again ignore unemployment and assume now that a job change is recorded only at the end of the year in which it occurs; X_n can, therefore, take on integral values only. This discrete approximation to the employment termination process is not likely to create unacceptable distortions of reality, especially in the aggregate, provided that the turnover rates for the groups of workers under consideration are not very high. For the assumption implies a forward displacement, by a fraction of a year in an individual realization, of all the intermediate points of employment termination, with partially compensating effects in terms of the successive lengths of service. Evidently, the structure can be refined by recording, for example, monthly changes at the expense of a twelve-fold increase in the state space and of difficulties in data collection.

Accordingly, an individual who currently has i years of service at his job next year will have either 0 years of service at a new job or i+1 years at his current job. If during the course of his working life he holds k jobs, and if job changes occur at the end of years $n_1, n_2, \ldots, n_k$, then the total number of pensionable years, $\alpha_{as}(N)$, under the composite vesting rule [a,s] will be

$$\alpha_{as}(N) = \sum_{j=1}^{k} Y_{n_j} \tag{22}$$

where

$$Y_{n_j} = \begin{cases} X_{n_j} + 1 & \text{if } n_j \geq a \text{ and } X_{n_j} \geq s-1 \\ \\ 0 & \text{otherwise.} \end{cases} \tag{23}$$

3.1. Development of the Model

To compute the expected value of $\alpha_{as}(N)$, it is sufficient to know the probabilities of the events associated with the right-hand side of (22), i.e., events of the type $\{X_n = 0, X_{n-1} = j\}$. For given n and j, such an event implies the addition of j+1 ($\geq$ s) years of service to the individual's pension at the beginning of his n-th ($\geq$ a) year of employment. Since the employment that terminates at the beginning of the n-th year with j+1 years of service must have started at the beginning of the n-(j+1)th year of employment ($n \geq j+1$), the above event is equivalent to the event $\{X_n = 0, X_{n-1} = j, X_{n-2} = j-1, \ldots, X_{n-j-1} = 0\}$. Therefore, we can write:

$$P[X_n = 0, X_{n-1} = j] = P[X_n = 0 | X_{n-1} = j] P[X_{n-1} = j | X_{n-j-1} = 0] P[X_{n-j-1} = 0]. \tag{24}$$

The conditional probabilities at the right-hand side of (24) are directly obtained from the data (see Appendix 2) while the last term can be derived recursively from

$$P[X_n = 0] = \sum_{i=0}^{n-1} P[X_n = 0 | X_{n-1} = n-i-1] P[X_{n-1} = n-i-1 | X_i = 0] P[X_i = 0],$$

$$n = 1,2,\ldots,N-1 \qquad (25)$$

with the (formal) initial condition $P[X_0 = 0] = 1$.

Based on the above observations, an algorithm for the computation of the expected number of pensionable years is presented in Appendix 2.

The approach being used in this section also makes it possible to compute the distribution function of the cumulative pensionable service over time, thus leading to a complete characterization. For this purpose, we denote by $W^n(i,k)$ the probability that by the end of his n-th year of employment, an individual has accumulated k pensionable years in his previous jobs and has i years of service in his present employment, all under the vesting rule [a,s]. $W^n(i,k)$ contains all the information that might be required as to the number of pensionable years and/or the years of service in the current employment at a given time. In particular, the distribution $Q_n(k)$, $k = 0,s,s+1,\ldots,n$ of the number of pensionable years by year n is given by

$$Q_n(k) = \sum_{j=0}^{s-2} W^n(j,k) + \sum_{j=s-1}^{n} W^n(j,k-j). \qquad (26)$$

A recursive algorithm for the computation of the probabilities $W^n(j,k)$ is also included in Appendix 2.

3.2. Application to Canadian Civil Service

The models discussed so far are capable of taking into account the dependency of the termination rates or completed lengths of service on both age and the duration of employment in a given job. In most private pension plans, however, data on termination rates is gathered only according to age. Although the models will also be applicable in these cases, their full potential can only be explored with a proper data base. Such a source, which will be used in this section for an application of the discrete model, is found in the Actuarial Examination of the Canadian Public Service Superannuation Account, as of December 31, 1972 [5]. Summary information on the data base is presented in Table 1. A comparative examination of this table indicated that the group selected is representative of a significant segment of the Canadian white collar work force.

Termination rates used in the calculations are those of male employees only and are given in Appendix 3. These rates are taken as the estimates of the probabilities p_i^n used as input to the model (see Appendix 2). For given i and n, p_i^n denotes the probability that the employment of an individual, who entered his job at his n-th working year, will terminate by the end of the year with i+1 years of service in the organization. Termination rates listed in Appendix 3 are the estimates of p_i^n for $i = 0,1,2,3,4$ and $n = 0,1,\ldots 39$; and for $i \geq 5$ and $5 \leq n+i \leq 43$ such that n+i+20 is the age at termination. $p_i^n = 1$ for $n+i = 44$, which implies a mandatory retirement age of 65.

TABLE 1: Employers in the Canadian Public Service and Terminations During 1968-72 (Source: Reference [5])

	Employers on Jan. 1 1968	Entrants During 1968-72	Terminations During 1968-72: Less than five years service	Terminations During 1968-72: Five or more years of service*	Employers on Dec. 31 1972
Males	160,970	88,750	41,780	25,944	181,996
Females	57,459	66,245	36,924	11,818	74,962
Total	218,429	154,995	78,704	37,762	256,958

* including retirement, death and disability

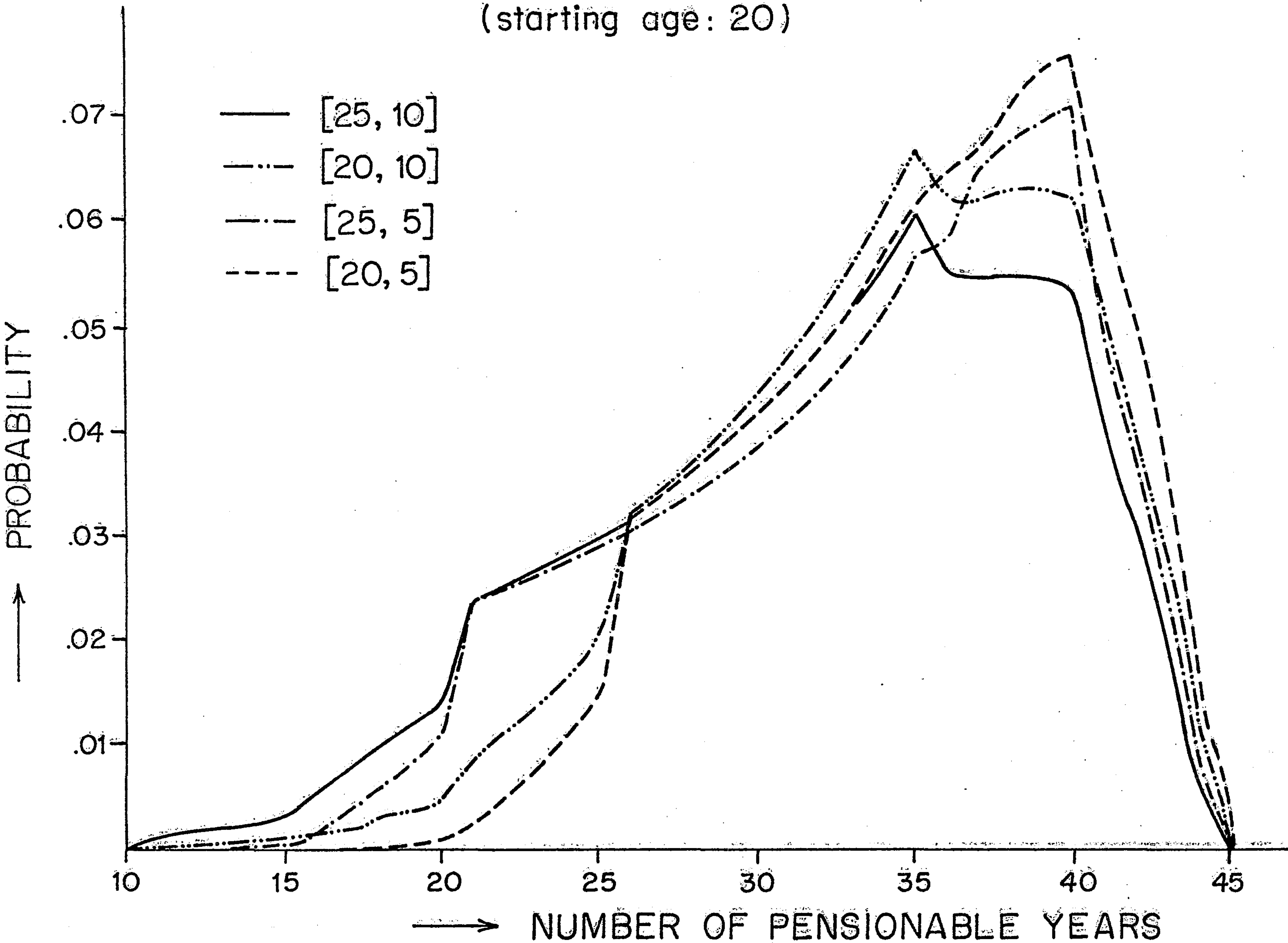
With the above data, we computed both the expectation and the distribution of the pensionable service, under different vesting rules, for groups of individuals entering the relevant segment of the Canadian labor market at different ages. Results for the starting age of 20 years are given in Table 2 and Figure 1. They appear to be relatively insensitive to variations in the vesting rules, apparently as a consequence of low termination rates that remain stable after the fifth year of service. This characteristic of the study group is typical of public servants and of comparable white-collar segments of the working population. Casual empiricism based on a study of the Economic Council of Canada leads us to very different results regarding the pension situation in other segments of the economy. Unfortunately, however, a national data base on termination rates by age and service is lacking, for a more comprehensive and comparative application,in the Canadian context.

TABLE 2: Expected Pensionable Years
As a Function of the Vesting Rules
(beginning age: 20)

MINIMUM YEARS OF SERVICE	MINIMUM AGE 25	30	35	40	45	50
2	42.8	40.4	38.0	36	34	32.3
3	42.5	40	37.8	35.7	33.8	32.1
4	41.9	39.7	37.4	35.4	33.4	31.8
5	41.2	39.3	37.1	35	33.1	31.4
6	40	38.7	36.5	34.5	32.6	31
8	38.5	38.1	36.2	34.3	32.4	30.8
10	37	37	35.7	33.9	32.1	30.5
12	35.6	35.6	35.1	33.5	31.8	30.2
15	33.5	33.5	33.5	32.6	31.1	29.6
20	29.8	29.8	29.8	29.8	29.2	27.9

FIGURE 1 : DISTRIBUTION OF PENSIONABLE SERVICE

(starting age: 20)



4. CONCLUSIONS

In constructing and solving the above models, we assumed for simplicity that all the employments offer pension plans (full coverage) with the same vesting rule. This situation is approximated in North America in industries associated with large, unionized firms. A United States Labor Department study shows, for example, that in April 1972 pension coverage in unionized firms was 91 percent [3]. In addition, competition and the collective bargaining process generally require that firms in a given segment of the economy have similar or identical vesting provisions.

However, pension coverage is not uniform throughout all segments of the North American economy. The study referred to above also indicates, for example, that the pension coverage in the U.S. retail trade industry was only 31 percent in 1972. An industry-wide application of the above models may, therefore, result in substantial overestimations unless partial coverage is taken into account.

At first sight, partial coverage in itself does not seem to complicate matters much. For if the probability of an employee's being in a covered employment is c, the expected pensionable service, for example, will simply be c times the expectation under full coverage. The problem becomes complicated if one wants to differentiate CLS distributions or turnover rates as between covered and noncovered employments. In addition, transferability (or portability), especially when coupled with partial coverage, would introduce further complications. Transferability refers to the percentage of covered employments from which <u>any</u> length of service can be transferred for pension purposes to another employment. These additional parameters must be taken into consideration, whenever they are pronounced, to approximate better the realities of a pension system.

Passage from pensionable service to pension benefits at retirement will be straightforward in certain circumstances. Consider, for example, an individual who becomes entitled to vested pensionable service at age 40 and expects to retire at 65. His pension benefits accruing from an employment are computed as a fraction of the wages paid to him during the last year of this employment times the number of pensionable years in this employment. Also, the pension benefits related to each employment, though not paid until retirement, are indexed from the time of termination. Under these circumstances, since wages from 40 to 65 years of age remain approximately constant, as a fraction of the average wage paid in the economy [4], the fraction of the percent wage times the expected pensionable service until retirement provides a good approximation of the expected purchasing power at retirement.

Unfortunately, many pension plans are not indexed, thus rendering this approximation invalid. One can, however, incorporate a wage function into the preceding models without much difficulty and investigate directly the accumulation over time of pension benefits.

The models presented in this paper provide analytical tools to forecast certain measures that are of interest to all the three parties directly or indirectly involved in pension plans: individuals, employers and governments. They also facilitate the comparison of different vesting rules through their consequences. Some of the possible extensions to the models include provisions for portability, partial coverage and nonuniform vesting. There is also a great need for empirical work in this area.

APPENDIX I

A1.1 Derivation of (7) and (8)

First, denote by $H(s,t)$ the expected number of pensionable employment terminations during $(0,t]$. We have

$$H(s,t) = \int_s^t h^*(u)du = G_0(t)-G_0(s)+\int_0^{t-s} h(u)[G_u(t-u)-G_u(s)]du\,. \tag{A1.1}$$

Next, let $\beta(s,t)$ be the expected cumulative pensionable service over $(0,t]$ arising from employments that terminate during this interval. That is, $\beta(s,t)$ is such that

$$\mu_s(t) = \begin{cases} \beta(s,t) + E[X(t)] & \text{if } X(t) \geq s \\ \\ \beta(s,t) & \text{if } X(t) < s \end{cases} \tag{A1.2}$$

where $X(t)$ is the backward recurrence time at time t. If Δs is a small increment in s, we can see that

$$\beta(s,t) - \beta(s-\Delta s,t) = (s-\Delta s)[H(s,t)-H(s-\Delta s,t)]+o(\Delta s)\,. \tag{A1.3}$$

This relation implies the differential equation

$$\begin{aligned} \frac{d}{ds}\beta(s,t) &= s\,\frac{d}{ds}H(s,t) \\ &= -s[g_0(s)+\int_0^{t-s} h(u)g_u(s)du],\ t \geq s\,, \end{aligned} \tag{A1.4}$$

which has the solution

$$\beta(s,t) = C - \int_{u=0}^{s} u[g_0(u)+\int_{x=0}^{t-u} h(x)g_x(u)dx]\,du,\ t \geq s \tag{A1.5}$$

where C is a constant with respect to s. On the other hand, we have

$$P[X(t) = t] = 1 - G_0(t) \tag{A1.6}$$

and, if $u_t(x)$ is the density related to $X(t)$,

$$u_t(x) = h(t-x)\,[1-G_{t-x}(x)],\ x < t. \tag{A1.7}$$

On combining the last three expressions in accordance with (A1.2) we obtain

$$\mu_s(T) = \beta(s,T) + T[1-G_0(T)] + \int_s^T x\,u_t(x)dx\,. \tag{A1.8}$$

If we use the initial condition

$$\mu_T(T) = T[1-G_0(T)] \tag{A1.9}$$

we find

$$C = \int_{u=0}^{T} u[g_0(u) + \int_{x=0}^{T-u} h(x)g_x(u)dx]\,du\,. \tag{A1.10}$$

The result (7) follows on substitution.

To prove (8) (and (13)), it is sufficient to note that

$$\alpha_{as}(T) = \begin{cases} \alpha_s(T) - \alpha_s(a) & \text{if } X(a) < s \\ \alpha_s(T) - \alpha_s(a) + X(a) & \text{if } X(a) \geq s\,. \end{cases} \tag{A1.11}$$

Since $\mu_s(t)$ is known, taking the expectations in the above using (A1.6) - (A1.7) for $t = a$, and after some rearrangements, (8) follows.

A 1.2 Derivation of (12) and of Higher Moments

The relation (11) implies

$$E[e^{-\theta\alpha_s(T)}] = [1-G(T)]e^{-\theta T} + \int_0^s E[e^{-\theta\alpha_s(T-u)}]g(u)du.$$
$$+\int_s^T e^{-\theta u}E[e^{-\theta\alpha_s(T-u)}]g(u)du \qquad \text{(A1.12)}$$

It follows, if $\mu_s^{(n)}(T)$ is the n-th moment, that

$$\mu_s^{(n)}(T) = T^n[1-G(T)] + \int_0^T \mu_s^{(n)}(T-u)g(u)du +$$
$$+ \sum_{j=1}^{n}\binom{n}{j}\int_s^T u^j\mu_s^{(n-j)}(T-u)g(u)du\,. \qquad \text{(A1.13)}$$

For n = 1, (A1.13) yields ($\mu_s^{(0)}(T) = 1$)

$$\mu_s^{(1)}(T) \equiv \mu_s(T) = T[1-G(T)] + \int_s^T ug(u)du + \int_s^T g(T-u)\,\mu_s(u)du\,. \qquad \text{(A1.14)}$$

This is a renewal equation whose solution is

$$\mu_s(T) = A_s(T) + \int_s^T A_s(u)h(T-u)du \qquad \text{(A1.15)}$$

where

$$A_s(t) = t[1-G(t)] + \int_s^T ug(u)du, \quad t \geq s\,. \qquad \text{(A1.16)}$$

(12) follows on substitution.

For the second moment, on using (12) in (A1.13) for n = 2, we find

$$\mu_s^{(2)}(T) = A_s^{(2)}(T) + \int_s^T A_s^{(2)}(u)h(T-u)du \qquad \text{(A1.17)}$$

where

$$A_s^{(2)}(t) = t^2[1-G(t)] + \int_s^t u^2 g(u)du + 2\int_s^{t-s}(t-u)g(t-u)\mu_s(u)du,$$
$$t \geq s. \qquad \text{(A1.18)}$$

Higher moments can also be obtained recursively.

APPENDIX 2

Algorithms for the Discrete Model

In terms of the probabilities p_i^n, the conditional probabilities on the right-hand side of (24) can be written, for $i = 0,1,\ldots,N-1$ and $n = i+1,\ldots,N$, as

$$P[X_n = 0 | X_{n-1} = i] = p_i^{n-i-1} \tag{A2.1}$$

and

$$P[X_{n-1} = i | X_{n-i-1} = 0] = \prod_{j=0}^{i-1} (1-p_j^{n-i-1}). \tag{A2.2}$$

We denote the product of these probabilities by g_i^n. As defined by (25), $p[X_i = 0]$ can now be computed recursively by

$$P[X_n = 0] = \sum_{j=0}^{n-1} P[X_j = 0]\, g_{n-j-1}^n, \quad n = 1,2,\ldots,N \tag{A2.3}$$

with $P[X_0 = 0] = 1$. Also, by (24), $P[X_n = 0, X_{n-1} = i]$, denoted by q_{i+1}^n, is given by

$$q_{i+1}^n = g_i^n\, P[X_{n-i-1} = 0]. \tag{A2.4}$$

q_{i+1}^n is the probability that at his n-th year of employment an individual terminates i+1 years of service with the firm. We can now construct, for $1 \leq n \leq N$, $1 \leq j \leq n$:

$$R_j^n = \sum_{k=j}^{n} k\, q_k^n \tag{A2.5}$$

and

$$\mu_{nj}(N) = \sum_{k=n}^{N} R_j^k, \tag{A2.6}$$

where R_j^n is the expected number of years in a job that terminates at the n-th year of employment with at least j years of service and $\mu_{nj}(N)$ is the expected number of pensionable years accumulated under the vesting rule [n,j].

In addition, $\sum_{n=0}^{N} P[X_n = 0]$ is the expected number of jobs occupied during one's working life and $\sum_{k=n}^{N} \sum_{i=j-1}^{k-1} q_i^k$ is the expected number of jobs that will lead to pension under the given vesting rule.

Also, under the composite vesting rule [a,s], the probabilities $W^n(i,k)$ that we introduced in section 3 can be computed recursively from the following, with the initial condition $W^0(0,0) = 1$.

$$W^{n+1}(i+1,k) = W^n(i,k)[1-p_i^{n-i}], \quad a-1 \le n \le N-1, s \le k \le n, \; 0 \le i \le n-k;$$

or:

$$0 \le n \le a-2, \; k = 0, \; 0 \le i \le n.$$

$$W^{n+1}(0,0) = \sum_{j=0}^{n} W^n(j,0)\, p_j^{n-j}, \quad 0 \le n \le a-2. \tag{A2.7}$$

$$W^{n+1}(0,k) = \sum_{j=0}^{\min(s-2,n)} W^n(j,k)\, p_j^{n-j} + \sum_{j=s-1}^{\max(n,s-2)} W^n(j,k-j-1) p_j^{n-j},$$

$$a-1 \le n \le N-1, \; s \le k \le n+1 (\text{or } k = 0)$$

$$W^{n+1}(i,k) = 0, \quad \text{otherwise.}$$

APPENDIX 3

Termination Rates (Source: Reference [5])

Age at beg. of service	Completed Years of Service: 0	1	2	3	4	≥ 5	Age at time of termination
20	.345	.171	.112	.088	.068		
21	.325	.166	.108	.085	.066		
22	.305	.161	.105	.082	.064		
23	.285	.156	.102	.079	.062		
24	.265	.151	.099	.076	.060		
25	.245	.146	.097	.073	.058	.037	25
26	.230	.141	.095	.070	.056	.035	26
27	.215	.137	.093	.068	.054	.034	27
28	.215	.133	.091	.066	.052	.033	28
29	.215	.129	.089	.064	.050	.031	29
30	.215	.125	.087	.062	.049	.030	30
31	.215	.122	.085	.060	.048	.028	31
32	.215	.120	.083	.059	.047	.027	32
33	.215	.119	.081	.058	.046	.025	33
34	.215	.118	.079	.057	.045	.024	34
35	.215	.117	.077	.056	.045	.023	35
36	.215	.116	.075	.056	.045	.022	36
37	.215	.115	.073	.056	.045	.020	37
38	.215	.114	.071	.056	.045	.019	38
39	.215	.113	.069	.056	.045	.019	39
40	.215	.112	.068	.055	.045	.018	40
41	.215	.111	.067	.055	.045	.017	41
42	.215	.110	.066	.055	.045	.016	42
43	.215	.109	.065	.055	.045	.015	43
44	.215	.108	.064	.055	.045	.014	44
45	.215	.107	.064	.054	.045	.012	45
46	.215	.106	.063	.054	.045	.011	46
47	.216	.105	.063	.054	.045	.010	47
48	.217	.104	.062	.054	.045	.009	48
49	.219	.103	.062	.054	.045	.009	49
50	.222	.102	.061	.053	.045	.008	50
51	.226	.101	.061	.053	.045	.008	51
52	.231	.100	.060	.053	.045	.009	52
53	.237	.100	.060	.053	.046	.009	53
54	.244	.101	.061	.054	.047	.009	54
55	.252	.102	.063	.056	.049	.010	55
56	.261	.104	.067	.060	.053	.010	56
57	.271	.107	.073	.066	.059	.010	57
58	.282	.111	.081	.074	.067	.010	58
59	.294	.116	.091	.084	.077	.010	59
						.150	60
						.090	61
						.095	62
						.115	63
						.160	64

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