#444-77

INSTITUTE FOR RESEARCH ON POVERTY DISCUSSION PAPERS

THE DYNAMICS OF LEARNING: A CONCEPTUAL MODEL

Maureen Hallinan and Aage B. Sørensen

UNIVERSITY OF WISCONSIN ~ MADISON

The Dynamics of Learning: A Conceptual Model

by

Maureen Hallinan
Stanford University
Stanford, California

Aage B. Sørensen
University of Wisconsin
Madison, Wisconsin

June 1977

Authorship is listed alphabetically. The research reported here was supported in part by Grant NIE-G-74-0047 to the Institute for Research on Poverty at the University of Wisconsin-Madison by the National Institute of Education, U.S. Department of Health, Education and Welfare. The opinions expressed are those of the authors.

ABSTRACT

This paper proposes a theory of the academic achievement process where learning is seen as the outcome of the utilization of opportunities for learning created in the teaching process by students characterized by a certain level of ability and effort. The theory is first formulated in a single equation model for learning where students' ability and effort is assumed constant over time. This model is then modified to take into account an interdependency between effort and achievement. Implications are derived of the proposed conceptions of the learning process for research on school effects and on inequality of educational opportunity.

The Dynamics of Learning: A Conceptual Model

INTRODUCTION

The primary objective of the school in contemporary society is to effect a particular kind of change, namely learning, in children. Closely related to this purpose is the school's goal of providing equal educational opportunities for all young people. To validly assess the success of schools in achieving these two ends requires an understanding of the learning process.

An attempt to conceptualize the learning process involves two efforts; first, to identify the variables relevant to learning, and second, to specify the way in which these variables interact to produce learning. Previous research has focused on one or the other of these two tasks. The tradition among sociologists has been to outline several individual and school level variables that are believed to affect achievement and to employ multivariate analytic techniques to determine the strength of these effects. Characteristics of students (ability, ethnicity, motivation), their families (SES, parents' education and occupation), peers (aspirations, ability), teachers (teaching style, educational level), and schools (expenditures, curriculum) have been associated with academic achievement. The Equality of Educational Opportunity Report (Coleman et al., 1966) is a classic example of sociologists' use of individual and school level variables to explain variance in student achievement. These achievement models are ordinarily static and assume that the association between the independent variables and achievement is the same at every point in time.

Developmental psychologists, on the other hand, have attempted to formulate learning theories that explain how select variables act to produce learning. Classical learning theorists have been concerned with the sequence in which the stimuli are presented for a learning event (Hilgard and Bower, 1966; Kimble, 1961), with contingency and reinforcement principles (Miller and Dollard, 1941), and with imitation (Bandura, 1971). Recently, researchers have concentrated on individual differences in learning, specifically, on learning rates (Carroll, 1963). Learning has been viewed as a process that is governed by student motivation and by the amount of time an individual student needs to learn a specific task (Bloom, 1971, 1973).

The Harnischfeger-Wiley model is the most recent conception of learning within this psychological tradition. This model is based on the assumption that a pupil's activities are central to his learning; a student learns a given subject to the extent that he spends time actively engaged in paying attention, studying, and trying to learn. The model relies on psychological learning theory to explain how actual learning time and the rate of learning affect the amount learned, and employs sociological variables such as institutional factors and teacher characteristics to explain the degree of a child's exposure to learning.

Since most sociological models of academic achievement are primarily static formulations that identify variables believed to influence a child's learning, they fail to specify the dynamic mechanisms underlying the relationships among these variables. Psychological learning theories, on the other hand, while describing how certain factors interact over time to produce learning, often fail to include variables generally believed to be key determinants of learning. There exists a need for a conceptualization of the learning process that is sufficiently comprehensive to include the relevant variables affecting learning and that, at the same time, can specify the mechanisms through which these variables interact over time to produce learning.

This paper presents a conceptualization of the learning process that sees learning as a process in which opportunities for learning presented by schools interact with characteristics of students to produce learning. We identify a set of basic concepts that describe learning and then formulate a model that depicts the relationships among these concepts. From this conceptualization and its formalization, we derive a set of implications for the interpretation of existing research and for the design of new research. Our present objective, therefore, is to outline a conceptual model; in future research the model will be tested on empirical data.

1. BASIC CONCEPTS

This section of the paper proposes a conception of the learning process and a simple mathematical model designed to mirror this conception. This model will be further modified and extended in subsequent sections. We will start with a quite elementary model to illustrate the basic ideas proposed in this paper.

There are important restrictions on the type of learning we will consider. First, we will be concerned only with learning that takes place in a formalized instructional setting as a result of the teaching process. Though some of our considerations may be relevant for other types of learning, we want to link teaching effort directly to learning. This means we will not be concerned with self learning, learning by trial and error, etc. In positive terms, the focus will be on learning that is a result of communication between a teacher and students where the teacher presents select instructional material to the students who in turn acquire some of the knowledge presented. Second, we will be concerned only with learning that can be registered on a test of academic achievement since this is the kind of learning that most schools try to produce.

The basic assumption made here is that learning produced by teaching is a function of (1) the amount of material presented to the students, (2) those variables that determine the students' ability to comprehend and retain the material presented and (3) those variables that affect a student's effort to learn. We refer to the first set of variables as opportunities for learning. These opportunities can be linked to characteristics of teachers, schools, and instructional organization. This is a critical variable in the learning process for no one will learn material he has not been exposed to, regardless of how much effort and ability the student displays. Previous studies have considered variables that are linked to opportunities for learning (such as various teacher characteristics and aspects of teacher behavior) but these variables ordinarily are introduced as independent variables in an additive model. This means that opportunities are seen as being able to compensate for the influence of ability and effort on learning. This conception is not an appropriate one, as we will argue below.

The second set of variables are those that determine a student's ability to utilize opportunities for learning. These variables have received considerable attention in both sociological and psychological research. Foremost among them is of course, IQ, but a number of other cognitive attributes, such as creativity and curiosity, have also been studied. In addition, part of what a student has already learned may be important for his ability to learn more. Hence, ability may be seen as having a constant and a variable component. A consideration of the variable component will be postponed to a later section of this paper.

Several personality attributes -- such as anxiety, need for achievement, level of aspiration, and attitudes toward learning -- have been suggested as influencing a student's effort to learn. These latter individual attributes are commonly introduced as independent variables in research, primarily, it seems, because they are assumed to produce variation in learning that reflects the variation in effort exhibited by a student with given cognitive skills.

In the next section we will attempt to justify a particular specification of the relationship among ability, effort, and opportunities for learning. Our goal is to avoid the failure of much existing research to recognize that learning is a change process, and that opportunities for learning ought to have a role in a model for learning.

2. A SINGLE EQUATION MODEL FOR LEARNING IN SCHOOLS

The preceding section has identified the basic concepts underlying learning and has argued for a dynamic model of the learning process. The objective of this section is to specify the fundamental mechanisms for change in learning over time by formulating a model that expresses the interrelationship among the basic concepts over time. The learning process will be described by a differential equation. The solution to this differential equation will provide the functional form for the interrelationship among variables measuring the basic concepts. The resulting model should be evaluated both for its conceptual content (that is, whether the model has properties that correspond to known features of learning) and for its empirical adequacy. Only the first task will be attempted here; this evaluation will lead to further modifications of the simple model proposed in this section.

The dependent variable in the model is learning as measured by some test of academic achievement. The amount learned at some time period t is denoted $y(t)$. The amount of learning that takes place in a small interval of time, dt, is represented by the change in $y(t)$ or $dy(t)$. The quantity $\frac{dy(t)}{dt}$ is the rate of learning which we will attempt to explain by opportunities for learning presented to the student and by his ability and effort.

The amount of knowledge and skills to which a teacher exposes a student determines that student's opportunities for learning. Call this amount of material $v(t)$. The amount of new material presented by a teacher in a small time interval dt is denoted $dv(t)$. Typically, a student learns only a fraction of the amount taught, the fraction depending on his ability and effort. Let s be a measure of the student's ability and effort, and assume for the present that s is constant over time. Then, for a single student, we have

$$dy(t) = sdv(t); \tag{1}$$

that is, the amount learned in a small time interval is linearly dependent on the student's ability and effort and the amount taught in that interval.

The solution to (1) is a simple linear expression $y(t) = sv(t)$ (assuming $y(0) = 0$) relating $y(t)$ and $v(t)$ for a single student. One may test this model and estimate parameters if a measure of $v(t)$ is available. However, it seems difficult to operationalize $v(t)$. An alternative strategy is to specify the dependency of $v(t)$ and $y(t)$ on time. This will allow a formulation of the model that does not necessitate direct measurement of $v(t)$, as shown below.

Assume that in a given subject, say algebra, there is a certain amount of knowledge, v^*, that represents the total amount of knowledge presented to the student, an amount ordinarily defined by a school curriculum. In some period of time, t, a portion $v(t)$ of v^* is presented to the student. The quantity $v^* - v(t)$ represents the amount not yet communicated by time t. When a teacher begins to teach a new subject, most of the material presented is unfamiliar to the students. As time goes on, the teaching process includes less presentation of new material and more repetition of material already taught. Toward the end of the unit, the amount of new material presented is likely to be small. It follows from these assumptions that the amount of new material taught depends on the quantity $v^* - v(t)$. This dependency can be formulated in the differential equation,

$$\frac{dv(t)}{dt} = k(v^* - v(t)) \qquad k > 0 . \tag{2}$$

Equation (2) specifies how much new material will be presented per unit time. It expresses the amount of effort displayed in the teaching process. It may be assumed that k is related to the total amount of material to be taught in a given subject; that is, k should reflect v^*. For simplicity we may set $k = 1/v^*$; that is, the more there is to teach, the more intense the teaching effort. This gives us a solution to (2):

$$v(t) = v^*(1 - e^{-\frac{1}{v^*} t}) = \frac{1}{k} (1 - e^{-kt}) \tag{3}$$

assuming $v(0) = 0$. This is the desired expression for the dependency of $v(t)$ on time.

We obtain a preliminary version of the model to be proposed by inserting (3) into the solution of equation (1):

$$y(t) = \frac{s}{k} (1 - e^{-kt}) . \tag{4}$$

This model generates the level of learning, or achievement, by time t as a product of the ability and effort of the student and the amount of material presented to the student by time t, i.e., the opportunities for learning. If no opportunities are presented, no learning will take place; as argued in the preceding section, this is a desired feature of the model.

The model as stated in (4) assumes that time is measured from the start of the teaching process. This is an awkward formulation for empirical applications. It will likely be the case that one obtains two or more observations on $y(t)$ where the first observations will not correspond to the start of the teaching process. An expression that will give the expected change in achievement over an arbitrary period of time is easily derived.

Suppose that observations are obtained for times t_1 and t_2, where $t_2 > t_1$ and both t_1 and t_2 are measured from the start of the teaching process. The corresponding values of $y(t)$ will be from (4):

$$y(t_1) = \frac{s}{k}(1 - e^{-kt_1}) \tag{5}$$

$$y(t_2) = \frac{s}{k}(1 - e^{-kt_2}) . \tag{6}$$

It follows that

$$\begin{aligned} y(t_2) - y(t_1) &= \frac{s}{k}(e^{-kt_1} - e^{-kt_2}) \\ &= \frac{s}{k} e^{-kt_1} (1 - e^{-k(t_2 - t_1)}) . \end{aligned} \tag{7}$$

The term e^{-kt_1} on the right hand side is still a problem since t_1 is measured from the start of the process. However, from (5) it follows that

$$e^{-kt_1} = 1 - \frac{k}{s} y(t_1) . \tag{8}$$

Substituting in (7) and rearranging gives,

$$y(t_2) = \frac{s}{k}(1 - e^{-k\Delta t}) + y(t_1)e^{-k\Delta t} \tag{9}$$

where $\Delta t = t_2 - t_1$. In the sequel (9) shall be used as our representation of the model and to simplify notation, time shall be measured from t_1 so that $t = \Delta t$. Further, equation (9) and its corresponding differential equation may be given a more familiar form by substituting $b = -k$ in (9). Differentiating then gives:

$$\frac{dy(t)}{dt} = s + by(t) \ . \tag{10}$$

This is a simple linear differential equation with a feedback that is negative in this application (since $b = -k < 0$). Hence, the model represents a learning process where the more that has already been learned, the less additional learning will take place. The negative feedback is produced by the opportunities for learning and will be larger the fewer opportunities for learning there are, i.e., the less material that is presented to the student (since $b = -1/v^*$). The parameter b, then, is a measure of opportunities for learning with a small b implying greater opportunities. The overall behavior of the system is such that the rate of learning will be high in the beginning of the teaching process and then gradually taper off, so that the student eventually will reach a maximum level of achievement, given his ability and effort and the opportunities for learning presented to him.

The behavior of the system seems reasonable in light of what learning curves are observed to be like in general. However, the model has only been formulated for a single student characterized by a given value of s. Furthermore, s is assumed to remain constant for the period of observation. The latter assumption will be modified in later sections of the paper; the former can easily be remedied here.

If the model is applied to a group of students, the quantity s presumably will vary among these students. This variation may be modeled by writing s as a function of a set of variables assumed to determine a person's ability and effort. The simplest such dependency would be a linear one. Hence, writing s as,

$$s = a_0 + a_1x_1 + a_2x_2 + . . . + a_nx_n \tag{11}$$

may capture the variation in s. Inserting this expression for s in equation (11) gives,

$$\frac{dy(t)}{dt} = a_0 + by(t) + a_1x_1 + a_2x_2 + . . . + a_nx_n . \tag{12}$$

The x_i variables would be both direct measures of a student's ability and effort such as the student's IQ score, and indirect measures such as the student's family background, etc. If it is assumed that the x_i vary across individuals but remain constant over time and that the parameters do not vary among students, the solution to (12) is,

$$y(t) = \frac{a_0}{b}(e^{bt} - 1) + y(0)e^{bt} + \frac{a_1}{b}(e^{bt} - 1)x_1 + . . . \tag{13}$$
$$+ \frac{a_n}{b}(e^{bt} - 1)x_n .$$

This is a lagged equation where the quantities $\frac{a_i}{b}(e^{bt} - 1)$ may be estimated as coefficients to the x_i variables, and the quantity e^{bt} may be estimated as the coefficient to y(0). From these estimates, the a_i and b parameters may be derived (see Sørensen and Hallinan, 1976). The lagged form of the equation poses estimation problems due to autocorrelation and measurement error. Methods of dealing with these technical problems are discussed at length in the econometrics literature and consequently will not be discussed here.

A variety of implications follow from this model, both with respect to the design of research and with respect to our understanding of the learning process. These implications will be discussed next. Then, a modification of the basic model will be undertaken to take into account the fact that one of the assumptions embodied in (12), namely, that the x_i variables remain constant over time, may not appear reasonable. The possible violation of this assumption does not affect the discussion that follows.

3. IMPLICATIONS

Since the conception of the learning process proposed here focuses on the learning of a specific body of material taught in schools, the time period of observation should not exceed the time period allocated to the subject in the classroom. Within this time frame, equation (12) may be estimated using observations of student achievement at two or more points in time and with appropriate measures of a student's ability and effort.

Estimates of the parameter b in the model, as indicated earlier, will provide a measure of the opportunities for learning in a classroom. Such measures may be used to study variations in opportunities for learning among classrooms and within a school as a function of teacher characteristics and classroom organization. Aggregating such measures across classrooms for a school will provide a measure of the opportunities for learning that characterizes a school. Variation among schools in opportunities for learning may be analyzed as a function of school characteristics, such as instructional expenditures, staffing policies, etc. We will first consider the implications of the conception proposed here for the study of school effects as brought about by variation in opportunities for learning. Then the implications for equality of opportunity are considered. We reserve for another paper discussion of the implications of the model for the study of classroom effects, since the advantages of the model for research in this area deserve extensive treatment.

School Effects

A large quantity of research and a great deal of debate have addressed the question of whether schools vary with respect to the amount of learning they produce in children. Much of the debate and research orginated with the EEO study (Coleman et al., 1966) which reported generally few and insignificant effects of schools; that is, almost all of the variation in achievement between schools could be accounted for by variation in children's family background and other personal characteristics. The only noteworthy effect of school variables was due to variation in the racial composition of classrooms presumably affecting the normative climate for learning in schools. This result ran counter to the expectation and intuition of many, particularly educators, who saw the findings as challenging the relevance of their efforts. The debate has since mainly addressed methodological issues -- both methodological problems in the original EEO report and general methodological problems raised by aggregating individual level observations to make school-level inferences.

The conception of learning proposed in this paper suggests several inadequacies in the research on school effects and provides new insights that may bring researchers closer to resolving the issue. Firstly, according to our model, schools may affect learning in two ways. Schools may vary in the opportunities for learning they provide and they may directly influence a student's ability and effort. The latter effect of schools can be analyzed only when ability and effort are allowed to vary which we permit in a subsequent section of this paper. For the present we concentrate on the opportunity effect of schools -- an effect that is determined by the instructional resources provided by schools. This is the impact of schools that presumably has been sought in much of the resarch on school effects.

Variation in the opportunities for learning among schools may be established using the proposed framework by adding the appropriate variables to equation (12). According to our model, the impact of opportunities for learning is an interactive one; that is, the opportunities for learning determine the influence of ability and effort on learning but do not add to their influence. All previous research on school effects has used linear models where school variables have been additively introduced as independent variables. This approach fails to capture the effect of schools on the ability and effort of students. While some experimentation with interaction effects has been attempted (Hauser et al., 1976), these tests are not adequate to detect the complex interactions proposed here.

A second consideration involves the kind of data used for an analysis of school effects. Most research on school effects uses cross-sectional data. Our model assumes that the opportunity effect of schools is reflected in change in academic achievement and cannot be identified by the level of achievement observed at a single point in time when the process is still ongoing. Longitudinal data are essential to capture opportunity effects on achievement. It is noteworthy that one of the four large scale studies that has focused on change in achievement (Summers and Wolfe, 1974) did find substantial effects of school characteristics on growth in academic achievement.[2]

The framework proposed here also implies that school effects are aggregates of the impact of opportunities for learning including characteristics of classrooms and of teachers and subject matter. Substantial within-school variation in opportunities for learning may co-exist with less strong between-school variation. To link characteristics such as teaching effectiveness to learning necessitates research at the appropriate level of analysis. The school effects research has almost always combined the individual and school level, and has ignored the classroom level where the immediate impact of the instructional resources of schools is to be found.

Finally, the considerations which led to the formulation of the model proposed here imply that school effects should be established on learning that takes place in schools. This is obvious, but it is a consideration that nevertheless is ignored in most sociological research on schools. When the impact of schools on learning is to be studied, it is of course necessary to have a measure of achievement that is comparable across schools. This has led to the adoption of general verbal ability and mathematical reasoning tests. But such measures are aptitude tests rather than tests of the amount of learning actually produced by schools. A student's aptitude is influenced by numerous sources making it difficult to isolate the influence of schools on change in such measures, especially when change is not directly focused upon.

Inequality of Opportunity

According to the learning model proposed here, opportunities for learning affect the observed impact of measures of ability and effort on learning. Formally, the observed coefficients to the x_i variables measuring a student's effort and ability (s) can be written (cf., equation 13):

$$d_i = \frac{a_i}{b} (e^{bt} - 1)x_i \qquad (14)$$

so that the observed d_i coefficients vary with b and t. A linear model for achievement that does not include the lagged term in (13) is generally used in cross-sectional research. This is a misspecification in that it assumes that the process is in equilibrium, since, as $t \to \infty$, d_i approaches $-\frac{a_i}{b} x_i$ and the lagged term of equation (13) drops out. The cross-sectional model thus assumes that the process of learning has been brought to completion. Even with the misspecification, the observed coefficients of the measures of ability and effort depend on b as they do in our model. This has some interesting implications for the assessment of inequality of opportunity in education.

Among the various x_i measures likely to be included in an analysis are measures of a student's family background; much research has been directed toward studying the impact of family background on learning in order to assess the degree of inequality of educational opportunity. It is often argued that "good" schools are schools where inequality of educational opportunity is reduced because students learn a great deal. This is faulty reasoning, as can be seen by our model.

According to equations (11) and (12), the impact of family background on learning is dependent on the effect of family background on a student's ability and effort and on the opportunities for learning presented to the student. The overall effect of family background on achievement then depends on b and on the coefficient a_i of a measure of family background in equation (14). For given a_i, the observed effect will increase as b is closer to zero. In other words, if the impact of family background on ability and effort is invariant across a set of schools, it is the good schools, i.e., schools that present students with a high level of opportunities for learning (b close to zero), that permits the greatest differential impact of family background and that produces the highest degree of inequality of opportunities. On the other hand, in schools where no one is given opportunities for learning, students with superior family background have no advantage. This can be illustrated directly from the model. If inequality is measured by the variance in y(t), it follows that when the learning process is completed, $\sigma^2(y(m)) = (-\frac{1}{b})^2\ \sigma^2(s)$ so that the higher the level of opportunities for learning the greater the amount of variance in achievement produced by a given variance in ability and effort, measured by the quantity s. Therefore, schools that provide a high level of opportunity for learning tend to increase inequality of opportunity. This implies that given the present structure of our schools, the goals of equality education and equality of educational opportunity may be in conflict.

The inferences about equality of opportunity were derived assuming that the contribution of family background to a person's ability and effort does not change over the schooling process, and that the variance in ability and effort is not affected either. As mentioned above, schools may have a direct impact on ability and effort. This relationship can only be analyzed in a model where ability and effort are allowed to change over time. In the next section we modify the unsatisfactory assumption made earlier that the individual determinants of learning remain constant over time.

4. EFFORT, ABILITY AND ACHIEVEMENT

Effort and ability are assumed constant in the model as formulated above. This is not a realistic assumption in the long run (that is, over extended periods of schooling) and it may not be realistic in the short run either, particularly with respect to effort.

There are two main sources of change in effort and ability. One source is those forces that, independent of the learning process, change the level of effort and ability a student will display. For example, a change in family environment (say a break-up of the family) may cause a student's motivation to change independent of what happens in the schooling process. Similarly, prolonged exposure to an intellectually unstimulating out-of-school environment may cause a decline in ability. The other source of change is the learning process itself. A student's effort may be influenced by his level of achievement and a student's ability to learn given material may depend on what has been learned in the past.

Change in ability and effort produced independent of the learning process is relatively easily incorporated in the model for learning. Rather than writing s as a constant in time, it should be written as a time dependent variable s(t). In order to estimate the resulting model, the dependency of s(t) on time should be specified. For relatively short periods of time between observations, a linear specification may suffice (see Coleman, 1968 for the resulting model). No fundamental modification of the model is needed; that is, a single equation model treating all variables except achievement as exogenous will suffice.

If change in ability and effort is produced by the learning process itself, it is another matter. Change in achievement (that is, learning) will produce change in the independent variables measuring ability and effort. The over-time course of the learning process cannot be inferred from equation (12), and estimation using equation (13) will produce biased estimates. A reformulation of the model is necessary to capture the interdependency among the variables ability, effort, and achievement. This resulting model will be a simultaneous equation model, and since we are concerned with change, a simultaneous differential equation model is most appropriate. The resulting complications are considerable. Some simplification may be obtained by considering the nature of change in ability and effort created by the learning process. This will be done next after which we shall return to the formulation of the model.

Ability and Learning

In section (1), ability was defined as referring to a student's cognitive skills (intelligence, etc.) in contrast to effort referring to the student's motivation to learn. Cognitive skills were assumed to be constant over time. This assumption is reasonable over short periods of time for such characteristics as intelligence, as it is usually conceived of. However, intelligence is only one of the components of a student's ability to learn. In particular, one may argue that a student's ability to learn depends on what has already been learned. One cannot learn history without some reading ability, or learn calculus without some knowledge of elementary algebra. If ability is conceived of in this broader sense, a student's ability to learn will depend on the learning process, making it necessary to model the interdependency between learning and ability.

Formally, the dependence of ability on learning can be expressed by writing ability as the sum of a constant and a variable component. The variable component refers to those skills acquired in previous learning. If ability is denoted $q(t)$, this quantity can be written as

$$q(t) = q(0) + q'(t) \tag{15}$$

where $q(0)$ denotes the constant component and $q'(t)$ denotes the variable component.

With this decomposition of the variable ability, the interdependency between learning and ability can be expressed as change in $q(t)$ being a function of learning. A linear relationship such that $q(t) = q(0) + cy(t)$ would be one possible specification. Inserting this expression in the model for learning given by equation (1) gives,

$$dy(t) = (s^* + cy(t))dv(t) \tag{16}$$

where s^* is a constant term that incorporates $q(0)$ and the level of effort, assumed to be constant. The solution to (16) is,

$$y(t) = \frac{s^*}{c}(e^{cv(t)} - 1) + y(0)e^{cv(t)} . \tag{17}$$

Inserting the expression for $v(t)$ gives the solution in time as

$$y(t) = \frac{s^*}{c}[\exp \frac{c}{b}(e^{bt} - 1) - 1] + y(0) \exp \frac{c}{b}(e^{bt} - 1) . \tag{18}$$

This expression models learning when ability to learn depends on the amount learned and on finite opportunities for learning. It is an expression similar in form to the one proposed before, but the time path of the learning process is more complicated. The quantity s can be written as a function of independent variables and the quantity s^*/c can be evaluated but it is not apparent how the parameters of the process can be identified.

The model just proposed is difficult to apply empirically. A more fundamental objection to the model derives from the assumption that ability to learn is linearly and continuously related to achievement. The assumption of linearity is made for convenience and lacks a substantive rationale. More seriously the assumption of continuity implies that everything learned will affect a student's ability to learn. This is not a realistic assumption. The schooling process exposes the student to a sequence of subjects such as algebra, geometry, and trigonometry. The learning of algebra may affect the ability to learn geometry, but within a given subject, the influence of what has been learned on what will be learned may be assumed to be modest. This means that a specification of the model which permits change in ability to take place after discrete time intervals (the completion of a curriculum unit) should be more appropriate.

These remarks suggest a solution to the problem of modeling the interplay between ability and learning; namely, to limit the time domain of the model to periods corresponding to curriculum subjects. These periods should be determined so that within a period learning does not depend strongly on what has already been learned. Between periods, such dependency will ordinarily exist, but the model will not capture this situation. In other words, by restricting the model to relatively short term variation in achievement the problem is avoided. This is consistent with the notion of opportunities for learning outlined earlier when opportunities for learning also were defined as specific to an instructional period.

While this design solution may overcome the problem associated with the interdependency between ability and learning, it will not overcome the problem of the interdependency of effort and learning. Here, short term variation is likely to exist. Further, the forces that generate such short term variation have considerable interest as reflecting the social organization of schools. These considerations and the resulting modifications of the model are discussed next.

Effort and Achievement

There are two ways that achievement is likely to affect effort to learn over time. In the first place, achievement is related to a set of external reinforcements given the child by both adults and peers. Teachers tend to praise successful students verbally and nonverbally (Rosenthal and Jacobson, 1968), interact with them frequently (Brophy and Good, 1970), assign them desired tasks and responsibilities (Flanders, 1960) and generally rank them high in the status hierarchy of the classroom (Hoehn, 1954). Parents often reward their child's academic achievement with material gifts as well as esteem and affection (Rosen and D'Andrade, 1959; Cervantes, 1965). Peers have been found to offer the high achiever respect and power (Gold, 1958; Glidewell, 1966). There is some evidence that high achievers are more popular with their classmates and assume more positions of leadership (Glidewell, 1966; Zander and Van Egmond, 1963). These positive reinforcements tend to motivate students, especially high achievers for whom rewards seem to be within reach, to study hard. Conversely, the reinforcements given the low achiever are often sanctions or punishments for their lack of academic accomplishments. While meant to increase a child's motivation, these punishments frequently produce the opposite effect by evoking in the student feelings of resentment or discouragement.

A second way in which academic achievement affects effort is through its impact on a child's self-concept. The school-aged child is in the process of learning about his capabilities and skills and many of his impressions of himself are formed by the image others reflect back to him during interaction (Mead, 1934). If a child is successful in school, teachers, parents, and students tend to communicate a respect for his achievement and a confidence that he can perform well (Rosenthal and Jacobson, 1968; Brophy and Good, 1970). This positive feedback shapes a student's image of himself as one who is capable of learning. One result of a strong self-image is to free a child from many of the anxieties about performance and the psychological obstacles to learning caused by feelings of inadequacy (Maccoby, 1966). The positive feedback given the higher achiever is likely to release the energy needed to continue trying to learn. At the same time, the successful child is experiencing satisfaction with himself and pride in his own accomplishments which create a desire to learn more. For the poor student, these same mechanisms are likely to result in a gradual loss of motivation to learn and withdrawal of effort.

An additional factor influences the motivation of the low achieving student. A child who experiences little or no success in school or who believes that he cannot learn seems to lose a sense of control over his environment (Wittes, 1970; St. John, 1972). Some students who have difficulty with school work tend to be fatalistic and to attribute their failure to an intractable environment which they cannot influence. Coleman et al., (1966) found considerable evidence of this attitude among disadvantaged low achievers. Loss of a sense of control, particularly over academic outcomes, is likely to be accompanied by a withdrawal of effort to learn.

These considerations illustrate the interdependence that exists between achievement and effort and suggest that this relationship is one of the most important mechanisms underlying the learning process. This interdependency needs to be taken into account in the model of the learning process. The strategy suggested in the discussion of the interdependency between ability and learning was to model ability to learn as a linear function of achievement. This strategy would not be suitable here. While it is conceivable that ability to learn may be continuously increasing with achievement, effort is more likely to fluctuate with definite boundaries. This suggests that change in effort should have a feedback on level of effort. Such a feedback, in combination within the effect of achievement on effort, needs to be modeled in a simultaneous differential equation framework. This task is attempted next.

5. A SIMULTANEOUS DIFFERENTIAL EQUATION MODEL FOR LEARNING

The previous section identified the need for a simultaneous differential equation model that would mirror the interdependency between learning and effort. Such a model is obtained by adding to equation (12) a second equation modeling change in effort, and by modifying (12) to take into account the time dependency of effort. A slight change in notation is useful. Denote by a_1 and a_2 the influence of exogenous variables, where the subscript indicates whether the term enters the achievement (19) or the effort (20) equation. Denote by $z(t)$ the level of effort at time t, and by c_1 and c_2 the effect of effort on achievement and achievement on effort respectively. Finally, denote by b_1 and b_2 the coefficients measuring the effect of the endogenous variables on themseleves, i.e., the feedback term.

With this notation the simplest linear system becomes:

$$\frac{dy(t)}{dt} = a_1 + b_1 y(t) + c_1 z(t) \qquad b_1 < 0,\ c_1 > 0 \tag{19}$$

$$\frac{dz(t)}{dt} = a_2 + b_2 z(t) + c_2 y(t) \qquad b_2 < 0,\ c_2 > 0\ . \tag{20}$$

The constant terms a_1 and a_2 may be written as functions of exogenous variables to capture variation in ability. This is an unnecessary complication here.

The system can be written in matrix form as,

$$\frac{dQ(t)}{dt} = A + DQ(t) \tag{21}$$

where $Q(t)$ is a vector with elements $y(t)$ and $z(t)$; A is a vector of constants a_1 and a_2, and D is a matrix of coefficients with elements b_1, b_2, c_1 and c_2. The solution to this system is,

$$Q(t) = A^* + Q(0)D^* \tag{22}$$

where $D^* = e^{Dt}$, and $A^* = AB^{-1}(e^{Dt} - 1)$.

In matrix notation, the solution to the system of differential equations is of the same form as (13). The equilibrium and stability conditions of the system are determined by the matrix of coefficients D. These conditions are often of major interest in studying systems of differential equations (see, for example, Richardson's 1960 model of Arms Races). However, in this instance, where the time domain of the system is restricted to the short run as a result of the considerations outlined in the discussion of the interdependency between ability and learning, there is empirically no reason to expect large fluctuations in achievement and effort. Instead, the major interest focuses on the interpretation of parameters, as was the case in the single equation model.

The interpretation of parameters should be derived from a consideration of the behavior of the system. For this purpose, it is useful to slightly reformulate the second equation (20) of the system. Denote by $\hat{y}(t)$ an expected level of achievement of a student at time t. This quantity may be assumed to reflect the expectations of significant others, particularly teachers. The variation in $\hat{y}(t)$ will be assumed to be exogenously determined, i.e., there is a set of expectations concerning what would be the proper level of achievement for a student that is determined by prior consideration and that remain constant over the time period being studied. Since $\hat{y}(t)$ is exogenous, this variable can be introduced into (20) without changing the basic properties of the system. Equation (20) then can be written,

$$\frac{dz(t)}{dt} = a_2 + b_1 z(t) + c_2[y(t) - \hat{y}(t)] \quad . \tag{23}$$

The reformulation expresses the hypothesis that it is not a student's absolute level of achievement that influences his effort, but the deviation of achievement from some standard. This standard may be common to all students if students are evaluated on universalistic criteria, or the standard may be specific to the student. In the latter case, y(t) should be treated as a variable not only varying over time but also between students in a fuller specification of the model.

With this reformulation, the parameter c_2 expresses how much the level of achievement -- relative to some standard or expectation -- influences change in effort. Achievement higher than $\hat{y}(t)$ will add to the change in effort; achievement lower than $\hat{y}(t)$ will have a negative impact and may -- depending on the values of other quantities enter (23) -- lead to a decrease in effort. Such a mechanism for change in effort will reflect the rewards students receive based on their academic performance. The magnitude of c_2 will therefore reflect the school (or rather classroom) reward structure. If c_2 is zero, achievement has no impact on effort, presumably because academic achievement is not a valued performance in the school. If c_2 is of large magnitude, achievement is very important for a person's effort, presumably because learning is highly valued, and students respond strongly to the rewards received (or not received) for their performance. This means that the parameter c_2 may be interpreted as a measure of the normative climate of schools -- an often sought school effect (e.g., McDill, Rigsby and Meyers, 1969).

The coefficient b_1 of equation (23) measures the feedback of change in effort on itself. This quantity is expected to be negative, since psychological and sociological constraints prevent effort from increasing or decreasing without limit. The amount of effort a student expends on learning, as discussed earlier, is influenced by the external reinforcements given by significant others. It is also influenced by an internal drive that has been described as a need for competence (White, 1959), or a need for achievement (Murray, 1938; McClelland, 1961). When a student is motivated primarily by external rewards, his effort depends more on the reward structure of the school and less on his own psychological need for achievement. In this case b_1 will be small in absolute value. When effort is only weakly dependent on the external reward system and the student is motivated primarily by internal needs for achievement, b_1 will be large. In this case, any deviation of the student's effort from its typical or equilibrium value will rapidly bring the level of effort back to its equilibrium. There may be social class and cultural variations in the degree to which motivation to learn is influenced by external rewards (Winterbottom, 1966; Rosen, 1956; Verhoff, 1960). We should therefore expect b_2 to vary across such family background groups.

Returning to equation (19), the coefficient b_1 again reflects opportunities for learning in the manner discussed earlier in this paper. It should be noted that if the single equation model is applied in situations where there is interdependency between effort and learning, estimates of the coefficient b will be biased measures of the level of opportunities for learning. This is because effort in the single equation model (12) is assumed constant. If change in effort occurs, it will be correlated with achievement according to equations (19) and (23); the relationship between the level of achievement and change in effort will inflate the b estimate and underestimate the opportunity effect of schools.

The parameter c_1 in the simultaneous equation model measures the effect of effort on learning. A small value of c_1 means that amount of effort has little impact on learning; a large value of c_1 indicates that effort makes a substantial contribution. We hypothesize that the value of c_1 will vary with subject matter reflecting variation in the cognitive structure of the instructional material taught in school. Learning history, for example, probably depends more on the amount of effort a student expends than learning mathematics which relies heavily on ability. The value of c_1 should be particularly large in subject where rote learning is demanded.

The simultaneous differential equation model identifies crucial concepts for the analysis of the learning process with the fundamental parameters b_1, b_2, c_1 and c_2. Of these, b_2 reflects the psychological structure of the motivation to learn, and c_1 reflects the cognitive structure of the subject matter, both measuring forces that are perhaps not easily modified by school organization and resource allocation. In contrast, b_1 and c_2 represent properties of the learning process that are subject to modification and therefore can be utilized in an attempt to influence the learning process. The parameter b_1 measures opportunities for learning, an important variable for the learning process that is subject to change by improving a school's instructional resources. The implications of this have already been discussed at length. The quantity c_2 represents another impact of schools on learning -- in this case, the normative climate of schools. This normative climate is modified by means other than the opportunities for learning; the social organization of schools should be especially important.

In discussing the implications of the single equation model, we pointed out that schools that provide a high level of opportunities for learning --other things equal -- will promote inequality in academic achievement and inequality of opportunity. These derivations were made assuming ability and effort of students constant over time, and assuming an invariant relationship between family background variables and the ability and effort of students. The simultaneous differential equation model proposed in this section shows that other things may not be equal; e.g., effort may vary over time as a reflection of the normative environment of schools. Schools may use this possibility of variation to equalize effort among students and to modify the influence of family background and thus counter the tendency toward inequality of opportunity outlined before. Such efforts must rely on measures of the potential impact of various strategies on the learning process. The sources of variation and their magnitudes should be assessed in a model that represents an adequate conception of the process. We believe that the model proposed here is a step in the direction of providing a more adequate model of the process.

REFERENCES

Bandura, A. Social Learning Theory. New York: General Learning Press, 1971.

Bloom, Benjamin. "Individual differences in school achievement: a vanishing point?" Bloomington, Indiana: Phi Delta Kappa, 1971.

_______________. "Time and learning." Thorndike Address, delivered at the Eighty-First Annual Convention of the American Psychological Association. Montreal, 1973.

Brophy, J. E., and T. L. Good. "Teachers' communication of differential expectations for children's classroom performance." Journal of Educational Psychology 61 (1970):365-374.

Carroll, J. B. "A model for school learning." Teachers College Record 64 (1963):723-733.

Cervantes, L. F. "The isolated nuclear family and the drop-out." Sociological Quarterly 6 (1965):103-118.

Coleman, James S. "The mathematical study of change," in H. M. Blalock and A. B. Blalock (ed.), Methodology in Social Research. New York: McGraw Hill, 1968, pp. 428-478.

_______________, Ernest Q. Campbell, Carol J. Hobson, James McPartland, Alexander M. Mood, Frederic D. Weinfeild, and Robert L. York. Equality of Educational Opportunity. Washington, D. C.: Office of Education, U. S. Department of Health, Education and Welfare, U. S. Government Printing Office, 1966.

Flanders, Ned A. Teacher Influence, Pupil Attitudes and Achievements. U. S. Department of Health, Education and Welfare, Office of Education, Cooperative Research Monograph No. 12, 1960.

Glidewell, John C., Mildred B. Kantor, Louis M. Smith, and Laverne A. Stringer. "Socialization and social structure in the classroom," in M. L. Hoffman and L. N. Hoffman (ed.), Review of Research in Child Development, No. 2. New York: Russell Sage Foundation, 1966, pp. 221-256.

Gold, M. "Power in the classroom." Sociometry 27 (1958):50-60.

Hauser, Robert M., William H. Sewell, and Duane F. Alwin. "High school effects on achievement," in W. H. Sewell, R. M. Hauser, and D. L. Featherman (ed.), Schooling and Achievement in American Society. New York: Academic Press, 1976, pp. 309-342.

Hilgard, E. R., and G. H. Bower. Theories of Learning. New York: Appleton-Century-Crofts, 1966.

Hoehn, A. J. "A study of social status differentiation in the classroom behavior of nineteen third grade teachers." Journal of Social Psychology 39 (1954):269-292.

Kimble, G. A. Hilgard and Marquis' Conditioning and Learning. New York: Appleton-Century-Crofts, 1961.

Maccoby, Eleanor (ed.). The Development of Sex Differences. Stanford: Stanford University Press, 1966.

McClelland, David. The Achieving Society. New York: Free Press, 1961.

McDill, E. L., L. C. Rigsby, and E. Meyers, Jr. "Educational climates of high schools: their effects and sources." American Journal of Sociology 74 (1969):567-586.

Mead, George H. _Mind, Self and Society_. Chicago: University of Chicago Press, 1934.

Miller, N. E., and J. Dollard. _Social Learning and Imitation_. New Haven: Yale University Press, 1941.

Murray, H. A. _Explorations in Personality_. New York: Oxford University Press, 1938.

Richardson, Lewis F. _Arms and Insecurity_. Pittsburgh: The Boxwood Press, 1960.

Rosen, B. C. "The achievement syndrome: a psychocultural dimension of social stratification." _American Sociological Review_ 21 (1956):203-211.

Rosen, B. S., and R. D'Andrade. "The psycho-social origin of achievement motivation." _Sociometry_ 22 (1959):185-217.

Rosenthal, Robert, and Lenore Jacobsen. _Pygmalion in the Classroom_. New York: Holt, Rinehart and Winston, 1968.

St. John, Nancy. "The elementary classroom as a frog pond: self concept, sense of control and social context." Unpublished manuscript, Harvard University, 1972.

Sørensen, Aage, and Maureen Hallinan. "A reconceptualization of school effects." Unpublished manuscript, University of Wisconsin, Madison, 1976.

Summers, A. A., and B. L. Wolfe. "Equality of educational opportunity quantified: a production function approach." Paper presented at the Economic Society Winter Meetings, December, 1974.

Veroff, J., et al. "Achievement motivation and religious background." American Sociological Review 27 (1962):205-217.

Winterbottom, Marian. "The relation of need for achievement to learning experiences in independence and mastery," in H. Proshansky and B. Seidenberg (ed.), Basic Studies in Social Psychology. New York: Holt, Rinehart and Winston, 1966, pp. 294-307.

White, Robert W. "Motivation reconsidered: the concept of competence." Psychological Review 66 (1959):317-318.

Wittes, S. People and Power: A Study of Crisis in Secondary Schools. Ann Arbor, Michigan: University of Michigan, 1970.

Zander, A., and E. Van Egmond. "Relationship of intelligence and social power to interpersonal behavior in children," in W. W. Charters, and N. Gage (ed.), Readings in Social Psychology of Education. Boston: Allyn and Bacon, 1963, pp. 114-123.